Generating functions involving binomial coefficients $\binom{4n}{2n}$ it's squared, reciprocal and their closed forms for hypergeometric expressions

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Abstract

In this paper we consider the infinite series pertaining to the binomial coefficients $\binom{4n}{2n}$ where we make study on the several classes of generating functions containing the coefficients, $\binom{4n}{2n}$, it's squared $\binom{4n}{2n}^2$ and it's reciprocal $\binom{4n}{2n}^{-1}$ by utilizing the generating function and integral representation of central binomial coefficients. Also we make discussions on generating functions of variants form of *main results* and we make an attempt to give closed forms for the respective hypergeometric forms.

Key words: central binomial coefficients, generating function, dilogarithm function, hypergeometric function.

1 Introduction

Central binomial coefficients are the particular type of positive integers of binomial coefficients that appear exactly in the middle of the even numbered rows of the Pascal triangle which we define them by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}, \quad \forall n \geq 0$$

Interestingly, with the binomial series expansion of the function $(1-4x)^{-1/2}$ for $|x| < 1/4$, the coefficients show up their presence providing us with generating function as

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots$$

The proof of the last result can easily be obtained in classical combinatorial books and a classical proof is discussed in [1] (see page 2, Lemma 1.1).

Since long time the study of central binomial coefficients is being made resulting intriguing identities and theorems in the field of number theory, combinatoric and calculus in the study of infinite series and integrals. Many intriguing results/power series were studied and discussed by Lehmer [2].

This paper is meant to be studying the coefficients of central binomial coefficients that appear at the even position namely, $1, 6, 70, 924, 12870, \cdots$, ie; the coefficient defined by

$$\binom{4n}{2n} = \frac{(4n)!}{((2n)!)^2}, \quad \forall n \geq 0$$

As we have the generating function of central binomial coefficients and with the aid of it, it is easy to deduce the generating function for the evened central binomial coefficients, $\binom{4n}{2n}$. In other words

$$\sum_{n=0}^{\infty} \binom{4n}{2n} x^n = \frac{1}{\sqrt{2}} \sqrt{\frac{1+\sqrt{1-16x}}{1-16x}}, \quad |x| < 1/16 \tag{1}$$

A proof is discussed in the article [1] (see page 4, Lemma 2.1).

2 Generating function

Now we consider five sequences pertaining to even central binomial coefficients with it's reciprocal and squared version for which we shall be deriving generating function and we define them as

$$\mathcal{A}_n = \binom{4n}{2n}, \quad \mathcal{B}_n = \frac{1}{n}\binom{4n}{2n}, \quad \mathcal{C}_n = \frac{1}{n^2}\binom{4n}{2n}, \quad \mathcal{D}_n = \binom{4n}{2n}^2, \quad \mathcal{E}_n = \binom{4n}{2n}^{-1}$$

Since we already know that for $|x| < 1/16$, $\sum_{n=0}^{\infty} \mathcal{A}_n x^n = \frac{1}{\sqrt{2}}\sqrt{\frac{1+\sqrt{1-16x}}{1-16x}}$, on dividing by x and integrating from 0 to z for $|z| \le 1/16$ we get

$$\sum_{n=1}^{\infty} \mathcal{B}_n z^n = 4\ln 2 - \ln\left(1+\sqrt{1-16z}\right) - 2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1-16z}}\right) \quad (2)$$

which is the strategy used in my article [1] (page 6) however, we shall find a different approach for (2) also the central notion of the paper for generating functions is heavily based on differention and integration method of the resulted power series. As the infinite sums are expressed in terms of hypergeometric function for which on actual solving became cumbersome to get the closed form in terms of elementary functions. So the auxiliary focus of the paper is too give possible elementry results to their respective hypergeometrics expression.

3 Theorems and Proofs

Theorem 3.1.(*First main result*) If $\mathcal{B}_n = \left\{\frac{1}{n}\binom{4n}{2n}\right\}_{n\ge 1}$, then for $|z| \le \frac{1}{16}$ the following equality holds.

$$\sum_{n=1}^{\infty} \mathcal{B}_n z^n = 4\ln 2 - \ln\left(1+\sqrt{1-16z}\right) - 2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1-16z}}\right) \quad (3)$$

Before we construct the proof of the theorem we need the Lemma required for the proof.
Lemma 3.1.1. For all $a, b > 0$, the following equality holds

$$\int_0^{\frac{\pi}{2}} \ln\left(a^2\cos^2 x + b^2 \sin^2 x\right) dx = \pi \ln\left(\frac{a+b}{2}\right)$$

Proof: The proof of lemma is mentioned in [1](see page no 3) which is based on the logarithmic series manipulation.

Proof of theorem 3.1. We make the use of *Wallis integral* [4], namely

$$\mathcal{W}_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{\pi}{2}\binom{2n}{n}\frac{1}{4^n} \quad \mathcal{W}_{4n} = \int_0^{\frac{\pi}{2}} \sin^{4n} x dx = \frac{\pi}{2}\binom{4n}{2n}\frac{1}{16^n}$$

rearranging the latter identity gives us $\binom{4n}{2n} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 16^n \sin^{4n} x dx$ and plugging the value of $\binom{4n}{2n}$ in (3) we get

$$\sum_{n=1}^{\infty} \mathcal{B}_n z^n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\sum_{n=1}^{\infty} \frac{(16z\sin^4 x)^n}{n}\right) dx = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log\left(1-16z\sin^4 x\right) dx$$

by factoring we observe that $1-16z\sin^4 x = (1+4\sqrt{z}\sin^2 x)(1-4\sqrt{z}\sin^2 x)$

$$\sum_{n=1}^{\infty} \mathcal{B}_n z^n = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(1+4\sqrt{z}\sin^2 x)dx - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \log(1-4\sqrt{z}\sin^2 x)dx$$

we can write $1 \pm 4\sqrt{z}\sin^2 x = \cos^2 x + \sin^2 x\sqrt{\left(1 \pm 4\sqrt{z}\right)^2}$ and by Lemma 3.1.1 it follows that

$$\sum_{n=1}^{\infty} \mathcal{B}_n z^n = 4\ln 2 - 2\underbrace{\left(\ln\left(1+\sqrt{1+4\sqrt{z}}\right) + \ln\left(1+\sqrt{1-4\sqrt{z}}\right)\right)}_{Q} \quad (4)$$

We further simply Q by simple mean of algebraic work and by rationalizing the expression gives

$$Q = 2\log\left(1+\sqrt{1-16x} + \frac{2(1+\sqrt{1-16z})}{\sqrt{1-4\sqrt{z}}+\sqrt{1+4\sqrt{z}}}\right)$$

by logarithmic properties it follows that

$$Q = 2\log\left(1+\sqrt{1-16z}\right) + 2\log\left(1+\left(\frac{4}{\left(\sqrt{1-4\sqrt{z}}+\sqrt{1+4\sqrt{z}}\right)^2}\right)^{1/2}\right)$$

$= \log(1+\sqrt{1-16z}) + 2\log\left((\sqrt{2})^2+\sqrt{2}\sqrt{1+\sqrt{1-16z}}\right) - \ln 2 =$

$\ln\left(1+\sqrt{1-16z}\right)$ $+2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1-16z}}\right)$ and hence combining $4\ln 2 - Q$ gives us the desired equality (3) and hence completes the proof.

Corollaries:
If $y = \text{sgn}(z)$ is the signum function for real value of $|z| \leq 1/16$, then

$$\sum_{n=1}^{\infty} (y)^n \mathcal{B}_n z^n = 4\ln 2 - \ln\left(1 + \sqrt{1 - y16z}\right) - 2\ln\left(\sqrt{2} + \sqrt{1 + \sqrt{1 - y16z}}\right)$$

Proof: The proof is trivial as it is nothing but just the replacing of x by $-x$ and vice-versa in (3) and at $x = 0$ we have $0 = 3\ln(2) - 2\ln\sqrt{8} = 0$ which is true.

Similarly,using (1) for $|x| < 1$ it is straightforward to deduce the following power series equalities

$$\sum_{n=0}^{\infty} \frac{y^n}{16^n} \binom{4n}{2n} \frac{x^n}{n+1} = \frac{8 - 2^{3/2}\left(1 + \sqrt{1 - xy}\right)^{3/2}}{3xy} \tag{5}$$

$$\sum_{n=0}^{\infty} \frac{y^n}{16^n} \binom{4n}{2n} \frac{x^n}{2n+1} = \frac{\sqrt{2\left(1 + \sqrt{1 - xy}\right)}}{\sqrt{yx}} = \frac{\sqrt{2}}{\sqrt{1 + \sqrt{1 - xy}}} \tag{6}$$

Proof: To prove (5) and (6), first we integrate (1) giving us (5) and further replacing x by x^2 and followed by integration of (1) yields the required equality (6).

Also subtracting twice of (6) from (5) give rise to generating function

$$\sum_{n=0}^{\infty} \frac{y^n \binom{4n}{2n} x^n}{16^n (2n+1)(n+1)} = \frac{6\sqrt{2}yx - \left(8 - 2^{3/2}\left(1 + \sqrt{1 - yx}\right)^{3/2}\right)\sqrt{1 + \sqrt{1 - yx}}}{3yx\sqrt{1 + \sqrt{1 - yx}}}$$

For (3) we now evaluate at some values of z, when $z = \pm\frac{1}{16\sqrt{5}}$ we can observe the appearance of golden ratio(ϕ) and it's reciprocal in the final closed form with the alternation of sign ie;

$$\sum_{n=1}^{\infty} \frac{1}{n} \binom{4n}{2n} \left(\frac{\pm 1}{16\sqrt{5}}\right)^n = 4\ln 2 - \ln\left(1 + \sqrt{\frac{2\phi^{\mp 1}}{\sqrt{5}}}\right) - 2\ln\left(\sqrt{2} + \sqrt{1 + \sqrt{\frac{2\phi^{\mp 1}}{\sqrt{5}}}}\right)$$

Also for $z=-\frac{1}{64}$ in (3) we get and identity in the form of golden ratio

$$\sum_{n=1}^{\infty}\frac{1}{n}\binom{4n}{2n}\frac{1}{64^n}=6\ln 2-\ln\left(1+2\phi\right)-2\ln\left(2+\sqrt{1+2\phi}\right)$$

Since the sum $\displaystyle\sum_{n=1}^{\infty}\mathcal{B}_n z^n=6{}_4F_3\left(1,1,\frac{5}{4},\frac{7}{4};\frac{3}{2},2,2;16z\right)z$ for which its corresponding simpler form is equal to (3) which in other words simplifies complex look it into simple result.

In the next section, we investigating the power series for $\frac{1}{n^2}\binom{4n}{2n}$. The work for desired power series is completely based on (3). Here $\mathrm{Li}_2(x)$ denotes the dilogarithm function which we will be encountering in course of the work.

Theorem 3.2.(*Second main result*) If $\mathcal{C}_n=\left\{\frac{1}{n^2}\binom{4n}{2n}\right\}_{n\geq 1}$ and for $|v|\leq\frac{1}{16}$ $\delta(v)=\sqrt{1+\sqrt{1-16z}}$, then the following equality holds for $\displaystyle\sum_{n=1}^{\infty}\mathcal{C}_n v^n$

$$=\mathcal{M}+2\mathrm{Li}_2\left(\frac{1}{2}-\frac{\delta(v)}{2\sqrt{2}}\right)+\mathrm{Li}_2\left(1-\frac{\delta^2(v)}{2}\right)+4\mathrm{Li}_2\left(-\frac{\delta(v)}{\sqrt{2}}\right)-\ln^2\left(\sqrt{2}+\delta(v)\right)+$$

$$4\ln 2\ln|v|-\frac{\ln^2\left(\delta^2(v)\right)}{2}-3\ln 2\ln\left(|\sqrt{2}-\delta(v)|\right)-\ln 2\ln\left(|2-\delta^2(v)|\right)-\ln 2\ln\delta^2(v)$$

where $\mathcal{M}$ is constant which is $2\zeta(2)+\dfrac{45}{4}\ln^2(2)$.

Proof of theorem 3.2: The proof can be proceed in the same way like that of theorem 3.1 however, we encountered integral $\displaystyle\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\mathrm{Li}_2(16v\sin^4 x)dx$ which become more complicated to solve and to develop the proof of it in an easy way we make use of the result (3).

Now dividing the power series obtained in (3) by z and on integrating from 0 to v gives us $\mathcal{I}(v)$

$$\sum_{n=1}^{\infty}\mathcal{C}_n v^n=\int_0^v\frac{4\ln 2-\ln\left(1+\sqrt{1-16z}\right)-2\ln\left(\sqrt{2}+\sqrt{1+\sqrt{1-16z}}\right)}{z}dz$$

It is easy to see that the integral has primitive in terms of dilogarithm and logarithmic functions and by applying the linearity of integral we see that primitive of it blows up at it's lower limit so we treat $\mathcal{I}(v)$ an an improper integral ie;

$$\lim_{\epsilon\to 0^+}\int_{\epsilon}^{v} f(z)dz = F(v) - \lim_{\epsilon\to 0^+} F(\epsilon) = F(v) - \mathcal{M}$$

where $f(z)$ is the integrand and $F(v)$ is an antiderivative. Now by linearity we see that

$$\mathcal{I}(v) = 4\ln 2\ln z - \underbrace{\int \frac{\ln(1+\sqrt{1-16z})}{z}dz}_{\mathcal{J}_1} - 2\underbrace{\int \frac{\ln(\sqrt{2}+\sqrt{1+\sqrt{1-16z}})}{z}dz}_{\mathcal{J}_2}$$

We evaluate $\mathcal{J}_1$ by making substitution $u = \sqrt{1-16z}$ and by partial fractions decomposition we have then

$$\mathcal{J}_1 = 2\int \frac{u\ln(1+u)}{u^2-1}du \overset{\text{PFD}}{=} \frac{\ln^2(1+u)}{2} + \int \frac{\ln(1+u)}{u-1}du$$

Now we set $u - 1 = w$ giving us

$$\int \frac{\ln(2+w)}{w}dw = \int \frac{\ln 2dw}{w} + \int \frac{\ln\left(1+\frac{w}{2}\right)}{w}dw = \ln 2\ln w - \mathrm{Li}_2\left(-\frac{w}{2}\right)$$

and hence combining the last two obtained primitive and making undo of each substitution we get $\mathcal{J}_1$ equal to

$$\mathrm{Li}_2\left(\frac{1-\sqrt{1-16z}}{2}\right) - \frac{\ln^2\left(1+\sqrt{1-16z}\right)}{2} - \ln(2)\ln\left(1-\sqrt{1-16z}\right) + C_1$$

In similar fashion, to evaluate $\mathcal{J}_2$ we substitute $\sqrt{1+\sqrt{1-16z}} = s$ and by making partial fraction of obtained result after substitution it yields $\mathcal{J}_2$

$$\int \frac{4(s^2-1)\ln\left(\sqrt{2}+s\right)}{s(s^2-2)}ds \overset{\text{PFD}}{=} \int \frac{\ln\left(s+\sqrt{2}\right)}{s^2-2}ds + 2\int \frac{\ln\left(s+\sqrt{2}\right)}{s}ds$$

as last three integrals are trivial with primitives $\displaystyle\int \frac{\ln\left(\sqrt{2}+s\right)}{s+\sqrt{2}}ds = \frac{\ln^2\left(s+\sqrt{2}\right)}{2}$ and with further substitution of $-s+\sqrt{2} = t$ and undoing the substitution we have

$$\int \frac{\ln\left(2\sqrt{2}-t\right)}{t}dt = -\frac{3}{2}\ln 2\ln\left(\sqrt{2}-s\right)+\mathrm{Li}_2\left(\frac{2-\sqrt{2}s}{4}\right)$$

and $2\int \frac{\ln\left(s+\sqrt{2}\right)}{s}ds = \ln 2\ln(s)-2\mathrm{Li}_2\left(-\frac{s}{\sqrt{2}}\right)$. Undoing the each substitution made gives us $\mathcal{J}_2$ equal to

$$-\ln 2\ln\left(\delta^2(v)\right)+2\mathrm{Li}_2\left(\frac{2-\sqrt{2}\sqrt{1+\sqrt{1-16z}}}{4}\right)+4\mathrm{Li}_2\left(-\frac{\sqrt{1+\sqrt{1-16z}}}{\sqrt{2}}\right)$$

$$-\ln^2\left(\sqrt{2}+\sqrt{1+\sqrt{1-16z}}\right)-3\ln 2\ln\left(\sqrt{2}-\sqrt{1+\sqrt{1-16z}}\right)+C_2$$

Thus on combining the results $4\ln 2\ln(z)-\mathcal{J}_1-\mathcal{J}_2$ and by fundamental theorem of calculus

$$\mathcal{I}(v) = F(v)-\lim_{\epsilon\to 0^+}\left(4\mathrm{Li}_2\left(-1\right)-\ln^2\left(2\sqrt{2}\right)-\frac{3\ln^2(2)}{2}-F(\epsilon)\right)$$

where $F(\epsilon) = 4\ln 2\ln\epsilon - 3\ln 2\ln\left(\sqrt{2}-\sqrt{1+\sqrt{1-16\epsilon}}\right) - \ln 2\ln\left(1-\sqrt{1-16\epsilon}\right)$
Now it enough to show that

$$\lim_{\epsilon\to 0^+}F(\epsilon) = -\lim_{\epsilon\to 0^+}\ln\left(\frac{\left(\sqrt{2}-\sqrt{1+\sqrt{1-16\epsilon}}\right)^3\left(1-\sqrt{1+16\epsilon}\right)}{\epsilon^4}\right)\ln 2$$

and with rationalization of the numerator and simplification gives us

$$-\lim_{\epsilon\to 0^+}\ln\left(\frac{16^4}{\left(1+\sqrt{1+\sqrt{16}\epsilon}\right)^4\left(\sqrt{2}+\sqrt{1+\sqrt{1+16\epsilon}}\right)}\right)\ln 2 = -\frac{15}{2}\ln^2 2$$

So

$$\mathcal{I}(v) = F(v)+2\zeta(2)+\frac{9}{4}\ln^2 2+\frac{1}{2}\ln^2 2+\frac{15}{2}\ln^2 2 = F(v)+\underbrace{2\zeta(2)+\frac{45}{4}\ln^2 2}_{\mathcal{M}}$$

where $F(v)$ is

$$\mathrm{Li}_2\left(\frac{1}{2}-\frac{\sqrt{1-16v}}{2}\right)-\frac{1}{2}\ln^2\left(1+\sqrt{1-16v}\right)-\ln 2\ln\left(\left|1-\sqrt{1-16v}\right|\right)$$
$$+4\ln 2\ln|v|+2\mathrm{Li}_2\left(\frac{1}{2}-\frac{\sqrt{1+\sqrt{1-16v}}}{2\sqrt{2}}\right)-\ln^2\left(\sqrt{2}+\sqrt{1+\sqrt{1-16v}}\right)$$
$$+4\mathrm{Li}_2\left(-\frac{\sqrt{1+\sqrt{1-16v}}}{\sqrt{2}}\right)-3\ln 2\ln\left(\left|\sqrt{2}-\sqrt{1+\sqrt{1-16v}}\right|\right)$$

and for convenience we write $\delta(v)=\sqrt{1+\sqrt{1-16v}}$ for $|v|\leq 1/16$ and combining $F(v)$ and $\mathcal{M}$ we get the desired result with the completion of proof of the theorem.

For the alternating version we replace v by $-v$ and the sum due to Wolfram alpha generates hypergeometric expression $6{}_5F_4\left(1,1,1,\frac{5}{4},\frac{7}{4};\frac{3}{2},2,2,2;16v\right)v$ which doesn't seems easy to be reducing to the *second main result*. However,the strategy above works effectively to provide elementary answer to it.

Corollaries:
Due to the theorem 3.2, we get some crazy results which are quite straightforward to show that following identities holds.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\binom{4n}{2n}\frac{1}{16^n}=\frac{7\pi^2}{12}-\frac{25}{4}\ln^2 2-4\mathrm{Li}_2\left(\frac{1}{\sqrt{2}}\right)+2\mathrm{Li}_2\left(\frac{2-\sqrt{2}}{4}\right)-\ln^2\left(1+\sqrt{2}\right)$$
$$+3\ln 2\ln\left(1+\sqrt{2}\right)\approx 0.5081222068\cdots \tag{7}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\binom{4n}{2n}\frac{1}{16^n}=\mathrm{Li}_2\left(\frac{1-\sqrt{2}}{2}\right)+4\mathrm{Li}_2\left(-\sqrt{\frac{1+\sqrt{2}}{2}}\right)+\frac{\pi^2}{3}-\frac{19}{4}\ln^2 2$$
$$+2\mathrm{Li}_2\left(\frac{\sqrt{2}-\sqrt{1+\sqrt{2}}}{2\sqrt{2}}\right)-\frac{\ln^2(1+\sqrt{2})}{2}-\ln^2\left(\sqrt{1+\sqrt{2}}+\sqrt{2}\right)$$
$$-3\ln 2\ln\left(\sqrt{1+\sqrt{2}}-\sqrt{2}\right)\approx -0.32379214\cdots. \tag{8}$$

As the main result boils down to the integral (see highlighted aforementioned result in red) and last two corollaries too provide the closed form for the integral in particular for $v = 1/16$ and $v = -1/16$ ie;

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\binom{4n}{2n}\left(\pm\frac{1}{16}\right)^n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{Li}_2\left(\pm\sin^4 x\right)dx = \begin{cases}(7)\,\text{if } + \text{ sign}\\ (8)\,\text{if } - \text{ sign}\end{cases}$$

And similarly, for any $|v| \leq 1/16$ we can now easily deduce the closed from for hypergeometric forms as well as for the aforementioned integral in red.

Since we discussed above on the generating function for $\frac{1}{n^2}\binom{4n}{2n}$ and following next section highlights it's light on the power series for squared of coefficients $\binom{4n}{2n}$ or $\binom{4n}{2n}^2$ in which work is accompanied by Elliptical integrals of first and second kind with their usual notations $K(x)$ and $E(x)$ respectively along with some intriguing identities.

Theorem 3.3. (*third main result*) If $\mathcal{D}_n = \left\{\binom{4n}{2n}^2\right\}_{n\geq 0}$ and for all $|w| < 1/256$, then the following equality holds.

$$\sum_{n=0}^{\infty} \mathcal{D}_n\left(\frac{w^2}{256}\right)^n = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{\frac{1+\sqrt{1-w\sin^4 y}}{1-w\sin^4 y}}dy = \frac{K\left(\sqrt{w}\right)-K\left(-\sqrt{w}\right)}{\pi}$$

where $K(x)$ is complete elliptical integral of the first kind.

Proof of theorem 3.3: The proof is constructed in a such way where we avoid the evaluation the integral appearing in the main result. To do so we now exploit the power series of $\mathcal{A}_n$ and integral representation of $\binom{4n}{2n}$ to obtained the desired integral.

$$\sum_{n=0}^{\infty}\binom{4n}{2n}^2\left(\frac{w}{256}\right)^n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{1}{16^n}\binom{4n}{2n}(w\sin^4 y)^n dy$$

and due to result (1) it follows that

$$\sum_{n=0}^{\infty}\binom{4n}{2n}^2\left(\frac{w}{256}\right)^n = \frac{\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\sqrt{\frac{1+\sqrt{1-w\sin^4 y}}{1-w\sin^4 y}}dy \tag{9}$$

And by the definition of complete elliptical integral of the first kind

$$K(w)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-w^2\sin^2\theta}}=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{1}{16^n}\binom{2n}{n}^2 w^{2n}, \tag{10}$$

Replacing w by $\sqrt{w}$ and $w\to-\sqrt{w}$. Adding series (9) at $\sqrt{w}$ and $-\sqrt{w}$ gives

$$\sum_{n=0}^{\infty}\binom{4n}{2n}^2\left(\frac{w^2}{256}\right)^n=\frac{K(\sqrt{w})+K(-\sqrt{w})}{\pi} \tag{11}$$

and from (9) and (11) we get the required result.
The last relation (11) corresponds to the hypergeometric expression

$$\sum_{n\geq 0}\mathcal{D}_n\left(\frac{w^2}{256}\right)^n={}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1;w\right)=\frac{K(\sqrt{w})+K(-\sqrt{w})}{\pi}$$

reducing the unpleasant and complex look of hypergeometric expression into simpler form of Elliptical integrals. Equation (11) now for any $|w|<1/256$ makes it possible to find the closed form in terms of elliptical form which are not nicer in look. So, we give some beautiful identities by the utility of (11).

Some intriguing identities and integral representation

In this section, we mention some intriguing identities on series and their corresponding integral representations involving the squared even central binomial coefficients $\binom{4n}{2n}^2$ by the explicit use of the theorem 3.3 and generating functions (5) and (6) respectively.

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(2n+1)}=\int_0^1\frac{K(\sqrt{w})+K(-\sqrt{x})}{\pi}dw=\frac{2}{\pi}-\frac{2\sqrt{2\pi}}{\Gamma^2\left(\frac{1}{4}\right)}+\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi\sqrt{2\pi}} \tag{12}$$

Proof: As we have already established the relation in (11) and we merely do integrate (11) within the interval of $w\in[0,1]$ yielding.

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(2n+1)}=\int_0^1\frac{K(\sqrt{w})+K(-\sqrt{w})}{\pi}dw \tag{13}$$

Equation (11) breaks down to the integral appearing in (14) and (15) respectively and by definition of complete elliptical integrals of first kind

$$\int_0^1\frac{d\theta}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{\sqrt{1-w\sin^2\theta}}+\frac{1}{\sqrt{1+w\sin^2\theta}}\right)dw=\frac{2}{\pi}+\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\sqrt{1+\sin^2\theta}}$$

For the fun purpose we handle (13) by the series manipulation, so by (10) and integrating we obtained the following series

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\pm 1)^n}{16^n(n+1)}\binom{2n}{n}^2=\int_0^{\frac{\pi}{2}}\left[\sum_{n=0}^{\infty}\frac{(\pm 1)^n}{4^n}\binom{2n}{n}\frac{\sin^{2n}\theta}{n+1}\right]d\theta \tag{14}$$

Using the generating function of central binomial coefficients (14) is easily deducible to

$$\int_0^{\frac{\pi}{2}}\left[\sum_{n=0}^{\infty}\frac{(\pm 1)^n}{4^n}\binom{2n}{n}\frac{\sin^{2n}\theta}{(n+1)}\right]d\theta=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin^2\theta}\int_0^{\sin^2\theta}\frac{dj}{\sqrt{1\pm j}} \tag{15}$$

Further with respective sign in (15),on integration it got reduced to two different integrals and adding them;

$$\int_0^{\frac{\pi}{2}}\frac{2-2\cos\theta}{\sin^2\theta}-2\int_0^{\frac{\pi}{2}}\frac{1-\sqrt{1+\sin^2 w}}{\sin^2 w}d\theta=2+\int_0^{\frac{\pi}{2}}\frac{2d\theta}{1+\sqrt{1+\sin^2\theta}} \tag{16}$$

Due integration by part it is easy to see

$$\int_0^{\frac{\pi}{2}}\frac{2d\theta}{1+\sqrt{1+\sin^2\theta}}=-2\int_0^{\frac{\pi}{2}}\frac{1-\sqrt{1+\sin^2\theta}}{\sin^2\theta}\theta\overset{\text{PFD}}{=}2\int_0^{\frac{\pi}{2}}\frac{1-\sin^2\theta}{\sqrt{1+\sin^2\theta}}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{4d\theta}{\sqrt{1+\sin^2\theta}}-2\int_0^{\frac{\pi}{2}}\sqrt{1+\sin^2\theta}d\theta=4K(-1)-2E(-1)$$

with $E(m)$ being *complete Elliptical integral of second kind.*
By standard definition of Elliptical integrals $4K(-1)-2E(-1)$ equates to

$$\int_0^1\frac{2-2u^2}{\sqrt{1-u^4}}du=\frac{1}{2}B\left(\frac{1}{4},\frac{1}{2}\right)-\frac{1}{2}B\left(\frac{3}{4},\frac{1}{2}\right)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}}-\frac{(2\pi)^{3/2}}{\Gamma^2\left(\frac{1}{4}\right)}$$

Dividing (16) by π and combining with last gives the required result.
From the above conclusion we also draw the following integral equality

$$\int_1^{\sqrt{2}}\frac{udu}{(1+u)\sqrt{(2-u^2)(u^2-1)}}=\frac{\pi}{2}\left(\frac{\Gamma^4\left(\frac{1}{4}\right)-8\pi^2}{(2\pi)^{3/2}\Gamma^2\left(\frac{1}{4}\right)}\right)=\frac{G\pi}{2}-\frac{1}{G\pi}$$

for the integral in (16) and we express the last identity in terms of constant called *Gauss Constant*,G.

Following the techniques used for (13) it is quite trivial to deduce the following elegant identities.

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(n+1)}=\frac{20}{9\pi}+\frac{\Gamma^2\left(\frac{1}{4}\right)}{9\pi\sqrt{2\pi}}+\frac{4\sqrt{2\pi}}{3\Gamma^2\left(\frac{1}{4}\right)} \tag{17}$$

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(2n+1)(n+1)}=\frac{16}{9\pi}-\frac{16\sqrt{2\pi}}{3\Gamma^2\left(\frac{1}{4}\right)}+\frac{4\sqrt{2}\Gamma^2\left(\frac{1}{4}\right)}{9\pi^{3/2}} \tag{18}$$

Relation (18) directly follows from (13) and (17). Interestingly, author noted the dazzling identities which the author mentioned as

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{2^{8n+3}(2n+1)^2}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(1+(-1)^n)}{16^n(n+1)^2}\binom{2n}{n}^2=\frac{1}{\pi}-\frac{\sqrt{2\pi}}{\Gamma^2\left(\frac{1}{4}\right)} \tag{19}$$

$$\sum_{n=0}^{\infty}\frac{1}{16^{n+1}}\left(\frac{\binom{4n}{2n}^2}{16^n(n+1)^2}-\frac{\binom{2n}{2n}^2}{2^{-1}(n+2)^2}\right)=\frac{2\sqrt{2\pi}}{3\Gamma^2\left(\frac{1}{4}\right)}-\frac{2}{9}+\frac{\sqrt{2}\Gamma^2\left(\frac{1}{4}\right)}{27\pi^{3/2}} \tag{20}$$

The validity and accuracy of the closed form has been confirmed by use of computer algebra system which are expressed in hypergeometric form.
Now employing (5) and (6) we give the some bewitching integrals form for (17) and (12) respectively.

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(2n+1)}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}\sin^{4n}\theta}{16^n(2n+1)}d\theta\overset{(6)}{=}\frac{2\sqrt{2}}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+\sqrt{1-\sin^4\theta}}}$$

$$\sum_{n=0}^{\infty}\frac{\binom{4n}{2n}^2}{256^n(n+1)}\overset{(5)}{=}\frac{4}{3\pi}\int_0^{\frac{\pi}{2}}\frac{\sqrt{2}\left(1+\cos\theta\sqrt{1+\sin^2\theta}\right)\sqrt{1+\sqrt{1-\sin^4 x}}-4}{\cos^2\theta(1+\sin^2\theta)-1}d\theta$$

These integrals are in agreement with (12) and (17) indeed via computer check.

Also for (18) we conclude an offbeat integral form

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{6\sqrt{2}\sin^4\theta-\left(8-2^{3/2}\left(1+\sqrt{1-\sin^4\theta}\right)^{3/2}\right)\sqrt{1+\cos\theta\sqrt{1+\sin^2\theta}}}{3\sqrt{1+\cos\theta\sqrt{1+\sin^2\theta}}}d\theta$$

which is merely the combination of last two integrals or generating function of sum in (18) with an output of $\dfrac{16}{9\pi} - \dfrac{16\sqrt{2\pi}}{3\Gamma^2\left(\frac{1}{4}\right)} + \dfrac{4\sqrt{2}\pi\Gamma^2\left(\frac{1}{4}\right)}{9\pi^{3/2}}$.

With the generating function of binomial coefficients $\binom{4n}{2n}^2$ our last focus will be on the generating function for the coefficient in reciprocal form which is now we make shed light on it in next section.

Theorem 3.4. (*fourth main result*) If $\mathcal{E}_n = \binom{4n}{2n}^{-1}$ and for all $|u| < 2$ the following equality holds

$$\sum_{n=0}^{\infty} \mathcal{E}_n u^{4n} = \frac{16}{16-u^4} + 2u\left(\frac{\arcsin\left(\frac{u}{2}\right)}{(4-u^2)^{3/2}} - \frac{\operatorname{arcsinh}\left(\frac{u}{2}\right)}{(4+u^2)^{3/2}}\right)$$

Proof of theorem 3.4: The notion of proof for the theorem has been provided in [3] but no closed form mentioned so we now be proving the result in general with an alternative way in which the idea of beta integral form of binomial coefficients is exploited. By beta integral form of binomial coefficients we have

$$\binom{n}{k}^{-1} = (n+1)\int_0^1 y^n(1-y)^n dy \Rightarrow \binom{4n}{2n}^{-1} = (4n+1)\int_0^1 y^{2n}(1-x)^{2n} dy$$

we multiply both sides by $u^{4n}(4n+1)$ and followed by summation

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}^{-1} u^{4n}}{4n+1} = \sum_{n=0}^{\infty} \int_0^1 u^{4n} y^{2n}(1-y)^{2n} dy = \int_0^1 \left(\sum_{n=0}^{\infty} u^{4n} y^{2n}(1-y)^{2n}\right) dy$$

we observed the elementary geometric series for all $|y| < 1$ in the latter result and hence by partial fraction decomposition (PFD) it follows

$$\int_0^1 \frac{dy}{1-u^4y^2(1-y)^2} \overset{\text{PFD}}{=} \frac{1}{2}\int_0^1 \left(\frac{dy}{1+u^2y(1-y)} + \frac{dy}{1-u^2y(1-y)}\right)$$

Last two integrals are standard arctangent intergrals which are trivial to show

$$\int_0^1 \frac{dy}{1-u^2y^2(1-y^2)} = \frac{2\tan^{-1}\left(\frac{u}{\sqrt{4-u^2}}\right)}{u\sqrt{4-u^2}} + \frac{2\tanh^{-1}\left(\frac{u}{\sqrt{4+u^2}}\right)}{u\sqrt{u^2+4}}$$

and since $\tan^{-1} y = \sin^{-1}\left(\frac{y}{\sqrt{1-y^2}}\right)$ and with $x \mapsto ix$ we get $\sinh^{-1} y = \tanh^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right)$ where $i = \sqrt{-1}$ that implies

$$\sum_{n=0}^{\infty} \binom{4n}{2n}^{-1} \frac{u^{4n}}{4n+1} = 2\left(\frac{\sin^{-1}\left(\frac{u}{2}\right)}{u\sqrt{4-u^2}} + \frac{\sinh^{-1}\left(\frac{u}{2}\right)}{u\sqrt{4+u^2}}\right) \tag{21}$$

We multiple by u and on differentaiting with respect to u gives us

$$\sum_{n=0}^{\infty} \binom{4n}{2n}^{-1} u^{4n} = \frac{2}{4-u^2} + \frac{2u\sin^{-1}\left(\frac{u}{2}\right)}{(4-u^2)^{3/2}} + \frac{2}{4+u^2} - \frac{2u\sinh^{-1}\left(\frac{u}{2}\right)}{(4+u^2)^{3/2}}$$

adding the results we obtained the desired equality and hence completes the proof.

The right hand expression of the theorem also suggests that the alternating series possess bizzare appearance of final result due to the involvement of complex numbers where the extraction of real part is complex to do however, for non alternating case it is pretty straightforward. Also we can observe that the hypergeometric form of the summation

$$\sum_{n\geq 0} \mathcal{E}_n u^{4n} = {}_3F_2\left(\frac{1}{2}, 1, 1; \frac{1}{4}, \frac{3}{4}; \frac{u^4}{16}\right), \quad |u| < 2$$

which is equal to the expression of right hand side of the theorem.

Corollaries: For $u = 1$ the following equality holds

$$\sum_{n=0}^{\infty} \binom{4n}{2n}^{-1} = \frac{16}{15} + \frac{\pi}{9\sqrt{3}} - \frac{2\log\phi}{5\sqrt{5}}$$

For the case of alternating series we perform $x \mapsto \sqrt{x}$ and then $x = \sqrt{-1}$ which gives ugly result in terms of complex numbers however, the outstanding simplified result is mentioned in [3] which agrees with the actual answer of the sum. Also we can obtained a squared power series of inverse sine and inverse hyperbolic sine.

$$4\sum_{n=0}^{\infty} \frac{16^n u^{4n+2}}{\binom{4n}{2n}(4n+1)(4n+2)} = \arcsin^2 u + \text{arcsinh}^2 u \tag{22}$$

The proof of the identity in (22) is easy to sketch as we merely need to transpose u to the left hand side of (21), hence on integrating and simplifying leads us the desired result.

Similarly, integrating (22) we further obtained the following taylor series for

$$4\sum_{n=0}^{\infty}\frac{16^n u^{4n+3}}{\binom{4n}{2n}(4n+1)(4n+2)(4n+3)}$$

$$=2\left(\sqrt{1-u^2}\arcsin u-\sqrt{1+u^2}\mathrm{arcsinh}u\right)+u\left(\arcsin^2 u+\mathrm{arcsinh}^2 u\right) \quad (23)$$

$u=1$ the sum attains the closed form

$$4\sum_{n\geq 0}\frac{16^n}{\binom{4n}{2n}(4n+1)(4n+2)(4n+3)}=\frac{\pi^2}{4}+\ln^2\left(1+\sqrt{2}\right)-2\sqrt{2}\ln(1+\sqrt{2})$$

which also leads to have simple closed form the complex hypergeometric

$$\frac{2}{3}{}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{5}{4},\frac{3}{2},\frac{7}{4},1\right)\approx 0.7513$$

which is numerically equals to the last result we obtained.

4 Theorems and proofs for some special cases

We present some exciting three identities deduced from (22) and (23).

Theorems: *The following two identities hold.*

$$4\sum_{n=0}^{\infty}\frac{16^n}{\binom{4n}{2n}(4n+1)^2(4n+2)}=4G+2\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right)-2\mathrm{Li}_2\left(-1-\sqrt{2}\right)-\frac{\pi^2}{12}$$

$$+\frac{\ln^2(2)}{4}-\ln^2\left(1+\sqrt{2}\right)-2\ln\left(1+\sqrt{2}\right)\ln\left(2+\sqrt{2}\right)$$

where G is *Catalan constant.*

$$4\sum_{n=0}^{\infty}\frac{16^n}{\binom{4n}{2n}(4n+1)(4n+2)^2}=\frac{\pi^2}{4}\ln 2-\frac{3}{8}\zeta(3)-\ln^2\left(1+\sqrt{2}\right)\ln\left(2+2\sqrt{2}\right)$$

$$+\frac{2}{3}\ln^3\left(1+\sqrt{2}\right)+\ln 2\ln^2\left(1+\sqrt{2}\right)+\frac{\mathrm{Li}_2\left(3+2\sqrt{2}\right)\ln\left(3+2\sqrt{2}\right)}{2}$$
$$-\frac{\mathrm{Li}_3\left(3+2\sqrt{2}\right)}{2}+\frac{i\pi\ln^2\left(3+2\sqrt{2}\right)}{4}+\frac{\ln^2\left(3+2\sqrt{2}\right)\ln\left(2+2\sqrt{2}\right)}{4}$$

Proofs of theorems: For the first identity we make use of (22) in which we divide both sides by u^2 and then integrating from 0 to 1 gives us

$$4\sum_{n=0}^{\infty}\frac{16^n}{\binom{4n}{2n}(4n+1)^2(4n+2)}=\int_0^1\frac{\arcsin^2 u+\mathrm{arcsinh}^2 u}{u^2}du$$

Applying the integration by parts in the latter expression we get

$$-\frac{\pi^2}{4}-\ln^2\left(1+\sqrt{2}\right)+2\int_0^1\frac{\arcsin u}{u\sqrt{1-u^2}}du+2\int_0^1\frac{\mathrm{arcsinh} u}{u\sqrt{1+u^2}}du$$

Subbing u by $\sin y$ and $\sinh u$ in former and latter intergral respectively.

$$2\int_0^{\frac{\pi}{2}}\frac{y}{\sin y}dy+2\int_0^{\ln\left(1+\sqrt{2}\right)}\frac{y}{\sinh y}dy=4G+4\int_0^{\ln\left(1+\sqrt{2}\right)}\frac{ye^y}{e^{2y}-1}dy$$

as the highlighted integral in red is well known result of *Catalan constant*,G and by substituting $y=\log t$ in the last integral then by partial fraction decomposition we obtained

$$4\int_0^{1+\sqrt{2}}\frac{\ln t}{t^2-1}dt\overset{\text{PFD}}{=}2\Big[-\mathrm{Li}_2(1-t)-\mathrm{Li}_2(-t)-\ln t\ln(t-1)\Big]_1^{1+\sqrt{2}}$$

$$\overset{\text{Definite integral}}{=}2\mathrm{Li}_2\left(-\frac{1}{\sqrt{2}}\right)-2\mathrm{Li}_2\left(-1-\sqrt{2}\right)+\frac{\pi^2}{12}-\ln(1+\sqrt{2})\ln(2+\sqrt{2})$$

where we employ the dilogarithm identity $\mathrm{Li}_2(-z)+\mathrm{Li}_2(-z^{-1})=-\zeta(2)-\frac{\ln^2(z)}{2}$ for $z=-\sqrt{2}$. Combining the obtained results gives the desired closed form.

For the second theorem we again explicitly make use of (22) where we divide both sides by u and carrying the integration from 0 to 1 yields

$$\int_0^1\frac{\arcsin^2 u+\mathrm{arcsinh}^2 u}{u}du\overset{\text{IBP}}{=}-2\int_0^1\frac{\arcsin u\ln u}{\sqrt{1-u^2}}du-2\int_0^1\frac{\mathrm{arcsinh} u\ln u}{\sqrt{1+u^2}}du$$

further by making substitution of u as $\sin y$ and $\sinh y$ in the aforementioned two integrals respectively gives rise to

$$-2\int_0^{\frac{\pi}{2}} y\ln(\sin y)dy - 2\int_0^{\ln(1+\sqrt{2})} y\ln\left(\sinh y\right)dy = \frac{\pi^2}{4}\ln 2 - \frac{7}{8}\zeta(3) + I$$

The red integral is straightforward by *Fourier series of* $\ln(\sin y)$ and I being the last integral which is our main focus of evaluation.

$$-2\int_0^{\ln(1+\sqrt{2})} \left(y\ln\left(e^{2y}-1\right) - y\ln(2e^y)\right)dy = -2\int_0^{\ln(1+\sqrt{2})} y\ln\left(e^{2y}-1\right)dy + \mathcal{V}$$

where $\mathcal{V} = \frac{2}{3}\operatorname{arcsinh}^3(1) + \ln 2\operatorname{arcsinh}^2(1)$ and for blue integral we perform IBP giving us

$$-\operatorname{arcsinh}^2(1)\ln\left(2+2\sqrt{2}\right) + 2\int_0^{\ln(1+\sqrt{2})} \frac{y^2e^{2y}}{-1+e^{2y}}dy = \mathcal{P} + \frac{1}{4}\int_1^{3+2\sqrt{2}} \frac{\ln^2 t}{t-1}dt$$

in fact the latter integral is easy to deduce by the magic of IBP.

$$\frac{1}{4}\left[\ln(1-t)\ln^2 t + 2\mathrm{Li}_2(t)\ln t - 2\mathrm{Li}_3(t)\right]_1^{3+2\sqrt{2}} = \frac{\zeta(3)}{2} - \frac{\mathrm{Li}_3\left(3+2\sqrt{2}\right)}{2}$$

$$+\frac{\ln^2\left(3+2\sqrt{2}\right)\ln\left(-2-2\sqrt{2}\right)}{4} - \frac{\mathrm{Li}_2\left(3+2\sqrt{2}\right)\ln\left(3+2\sqrt{2}\right)}{2}$$

since $\ln(-2-2\sqrt{2}) = i\pi + \ln(2+2\sqrt{2})$ and collecting the values of $\mathcal{P}$ and $\mathcal{V}$ for I we acquire the required closed form.

Theorem: *The following equality holds.*

$$4\sum_{n=0}^{\infty} \frac{16^n}{\binom{4n}{2n}(4n+1)(4n+2)(4n+3)^2} = 4G - \operatorname{arcsinh}(1)\left(4\sqrt{2} + \log\left(3-2\sqrt{2}\right)\right)$$

where G being *Catalan constant* and this result is acquired from (23). The integrals to be evaluated are trivial and in the final closed answer we come across a dilogarithm expression $\mathrm{Li}_2(1-\sqrt{2}) - \mathrm{Li}_2(-1+\sqrt{2})$ which is equal to $-\frac{\pi^2}{8} + \frac{1}{2}\ln^2(\sqrt{2}-1)$, evaluated by author himself in MSE (here).

The first identity on performaning partial fraction can be written into two series whose equivalent hypergeometric expression turns out to be

$$4{}_4F_3\left(\frac{1}{4},\frac{1}{2},1,1;\frac{3}{4},\frac{5}{4},\frac{5}{4};1\right)-2{}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{3}{4},\frac{5}{4},\frac{3}{2};1\right)\approx 2.11022$$

which attains the closed form derived for the aforementioned first identity. Similarly, second and third identity shows the heavy weight of hypergeometric expressions whose closed we have deduced easily by the mean of generating functions.

$${}_5F_4\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,1;\frac{3}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2};1\right)\approx 1.09551$$

$$\frac{2}{3}{}_4F_3\left(\frac{1}{2},\frac{1}{2},1,1;\frac{5}{4},\frac{3}{2},\frac{7}{4};1\right)-\frac{4}{9}{}_4F_3\left(\frac{1}{2},\frac{3}{4},1,1;\frac{5}{4},\frac{7}{4},\frac{7}{4};1\right)\approx 0.2317$$

For the justification of the results we make the use of CAS and the outputs are found be correct.

5 Conclusion

From the above study we give possible closed forms to the hypergeometric expression of heavy weight by introducing the powerful notion of generating function producing their correspondening interesting and useful results.

6 References

[1] N.Bhandari. Evaluating logarithmic integrals by the logarithmic series manipulation. *Romanian Mathematical Magazine,An Interactive Journal* https://www.ssmrmh.ro/2020/11/14/logarithmic-integrals-by-logarithmic -series/.

[2] D.H.Lehmer. Interesting series involving central binomial coefficients. *American Math, Monthly*,7-449-457, 1985.

[3] R. Sprugnoli. Sum of reciprocals of the central binomial coefficients. *Electronic Journal of Combinatorial Number Theory 6*, 2016.

[4] E.W.Weisstein. Wallis Cosine Formula.*MathWorld*.https://mathworl. wolfram.com//WallisCosineFormual.html/.