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ABOUT AN INEQUALITY BY NGUYEN VAN CANH-XII

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1) In ΔABC the following relationship holds:

$$\sum \sqrt{\tan\frac{B}{2}\tan\frac{C}{2}} \le \sqrt{\frac{3}{2}\sum \frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}$$

Nguyen Van Canh – Vietnam

Solution: We prove: **Lemma 1:**

2) In ΔABC the following relationship holds:

$$\sum \sqrt{\tan\frac{B}{2}\tan\frac{C}{2}} = \sum \sqrt{\frac{s-a}{s}}$$

Proof: Using $\tan\frac{B}{2}\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{s-a}{s}$, we obtain Lemma 1.We prove:

Lemma 2:

3) In ΔABC the following relationship holds:

$$\sum \frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 2$$

Proof: Using $\sum \frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \sum \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}} = \sum \frac{a}{s} = 2$,

we obtain Lemma 2.Let's get back to the main problem. Using the above Lemmas it suffices to prove that:

$\sum \sqrt{\frac{s-a}{s}} \le \sqrt{\frac{3}{2}\cdot 2} \Leftrightarrow \sum \sqrt{s-a} \le \sqrt{3s}$, true from CBS inequality:

$\left(\sum 1\cdot\sqrt{s-a}\right)^2 \le \sum 1 \sum(s-a) = 2s = \left(\sqrt{3s}\right)^2$, with equality for $a=b=c$.

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\sum \sqrt{\cot\frac{B}{2}\cot\frac{C}{2}} \le \prod \cot\frac{A}{2} \le 3\sqrt{3}\cdot\frac{R}{2r}$$

Nguyen Van Canh – Vietnam

Solution: We prove: **Lemma 1:**

5) In ΔABC the following inequality holds:

$$\sum \sqrt{\cot\frac{B}{2}\cot\frac{C}{2}} = \sqrt{\frac{s}{s-a}}$$

Proof: Using $\cot\frac{B}{2}\cot\frac{C}{2} = \sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s}{s-a}$, we obtain Lemma 1. We prove:

Using $\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} = \frac{s^2}{S} = \frac{s}{r}$

Let's get to the main problem.Using the above Lemmas it suffices to prove that:

$$\sum\sqrt{\frac{s}{s-a}} \le \frac{s}{r} \Leftrightarrow \sum\sqrt{\frac{s}{s-a}} \le \frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}} \Leftrightarrow \sum\sqrt{(s-b)(s-c)} \le s$$

true, from CBS inequality:

$\left(\sum\sqrt{(s-b)(s-c)}\right)^2 \le \sum(s-b)\sum(s-c) = s\cdot s = s^2$, with equality for $a=b=c$.

Equality holds if and only if the triangle is equilateral.

Inequality $\prod\cot\frac{A}{2} \le 3\sqrt{3}\cdot\frac{R}{2r} \Leftrightarrow \frac{s}{r} \le 3\sqrt{3}\cdot\frac{R}{2r}$, which follows from Mitrinovic's inequality:

$$s \le \frac{3R\sqrt{3}}{2}$$

Equality holds if and only the triangle is equilateral.

Reference:

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