R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-XXI

By Marin Chirciu-Romania

Edited by Florică Anastase-Romania

1) In ΔABC the following relationship holds:

$$\frac{m_a w_a}{a^2}+\frac{m_b w_b}{b^2}+\frac{m_c w_c}{c^2}\ge\frac{9}{4}$$

Proposed by Marian Ursărescu-Romania

Solution. Lemma. **2) In ΔABC the following relationship holds:**

$$m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$$

M.Lascu(1987)

Proof. Using identities $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and $\cos^2\frac{A}{2}=\frac{s(s-a)}{bc}$ inequality becomes as:

$$m_a\ge\frac{b+c}{2}\cos\frac{A}{2}\Leftrightarrow m_a^2\ge\left(\frac{b+c}{2}\right)^2\cos^2\frac{A}{2}\Leftrightarrow\frac{2b^2+2c^2-a^2}{4}\ge\left(\frac{b+c}{2}\right)^2\cdot\frac{s(s-a)}{bc}$$

$$\Leftrightarrow\frac{2b^2+2c^2-a^2}{4}\ge\frac{(b+c)^2}{4}\cdot\frac{(a+b+c)(b+c-a)}{4bc}$$

$$\Leftrightarrow 2b^2+2c^2-a^2\ge(b+c)^2\cdot\frac{(b+c)^2-a^2}{4bc}$$

$$\Leftrightarrow 4bc(2b^2+2c^2-a^2)\ge(b+c)^2\cdot[(b+c)^2-a^2]\Leftrightarrow a^2(b-c)^2\ge(b-c)^4$$

$\Leftrightarrow (b-c)^2(a-b+c)(a+b-c)\ge 0$. Equality holds for $a=b=c$.

Let's get back to the main problem. Using Lemma and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$E=LHS=\sum_{cyc}\frac{m_a w_a}{a^2}\ge\sum_{cyc}\frac{\frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cos\frac{A}{2}}{a^2}=\sum_{cyc}\frac{bc}{a^2}\cos^2\frac{A}{2}=$$

$$=\sum_{cyc}\frac{bc}{a^2}\cdot\frac{s(s-a)}{bc}=s\sum_{cyc}\frac{s-a}{a^2}=s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}=$$

$$=\frac{(16R-5r)(4R-3r)+r(4R+r)}{16R^2}=\frac{64R^2-64Rr+16r^2}{16R^2}=\frac{4R^2-4Rr+r^2}{16R^2}=$$

$$=\frac{(2R-r)^2}{R^2}=\left(2-\frac{r}{R}\right)^2\overset{Euler}{\ge}\frac{9}{4}=RHS.$$

Equality holds if and only if triangle is equilateral.**Remark.** Inequality can be much stronger.

3) In ΔABC the following relationship holds:

$$\frac{m_a w_a}{a^2}+\frac{m_b w_b}{b^2}+\frac{m_c w_c}{c^2}\ge\left(2-\frac{r}{R}\right)^2$$

Marin Chirciu

Solution. Using Lemma and $w_a=\frac{2bc}{b+c}\cos\frac{A}{2}$, we obtain:

$$E=LHS=\sum_{cyc}\frac{m_a w_a}{a^2}\ge\sum_{cyc}\frac{\frac{b+c}{2}\cos\frac{A}{2}\cdot\frac{2bc}{b+c}\cos\frac{A}{2}}{a^2}=\sum_{cyc}\frac{bc}{a^2}\cos^2\frac{A}{2}=$$

$$=\sum_{cyc}\frac{bc}{a^2}\cdot\frac{s(s-a)}{bc}=s\sum_{cyc}\frac{s-a}{a^2}=s\cdot\frac{s^2(s^2+2r^2-12Rr)+r^3(4R+r)}{16R^2r^2s}=$$

$$=\frac{(16R-5r)(4R-3r)+r(4R+r)}{16R^2}=\frac{64R^2-64Rr+16r^2}{16R^2}=\frac{4R^2-4Rr+r^2}{16R^2}=$$

$$=\frac{(2R-r)^2}{R^2}=\left(2-\frac{r}{R}\right)^2=RHS.$$

Equality holds if and only if triangle is equilateral.**Remark.** Inequality 3) is much stronger such inequality 1).

4) In ΔABC the following relationship holds:

$$\frac{m_a w_a}{a^2}+\frac{m_b w_b}{b^2}+\frac{m_c w_c}{c^2}\ge\left(2-\frac{r}{R}\right)^2\ge\frac{9}{4}$$

Marin Chirciu

Solution. See inequality 3) and $\left(2-\frac{r}{R}\right)^2\ge\frac{9}{4}\Leftrightarrow R\ge 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral.**Remark.** Let's find an opposite inequality.

5) In ΔABC the following relationship holds:

$$\frac{m_a w_a}{a^2}+\frac{m_b w_b}{b^2}+\frac{m_c w_c}{c^2}\le\frac{9}{4}\left(\frac{R}{2r}\right)^3$$

Marin Chirciu

Solution. Using $w_a\le m_a$, we get:

$$\sum_{cyc}\frac{m_a w_a}{a^2}\le\sum_{cyc}\frac{m_a^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(12R^2+8Rr-r^2)-r^3(4R+r)^3}{16R^2r^2s^2}=$$

$$=\frac{1}{16R^2r^2}\left[s^4+s^2(r^2-12Rr)+r^2(12R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]=$$

$$=\frac{1}{16R^2r^2}\left[s^2(s^2+r^2-12Rr)+r^2(12R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{s^2}\right]\overset{Gerretsen}{\leq}$$

$$\leq\frac{1}{16R^2r^2}\left[(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+r^2-12Rr)+r^2(12R^2+8Rr-r^2)-\frac{r^3(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right]=$$

$$=\frac{1}{16R^2r^2}\left[(4R^2+4Rr+3r^2)(4R^2-8Rr+4r^2)+r^2(12R^2+8Rr-r^3)-\frac{2r^3(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{1}{16R^2r^2}\left[16R^4-16R^3r+8R^2r^2+11r^4-\frac{2r^3(2R-r)(4R+r)}{R}\right]=$$

$$=\frac{1}{16R^2r^2}\cdot\frac{R(16R^4-16R^3r+8R^2r^2+11r^4)-2r^3(2R-r)(4R+r)}{R}=$$

$$=\frac{16R^5-16R^4r+8R^3r^2-16R^2r^3+15Rr^4+2r^4}{16R^3r^2}\overset{(2)}{\leq}\frac{9R^3}{32r^3}=\frac{9}{4}\left(\frac{R}{2r}\right)^3=RHS$$

$$\text{where, }(2)\Leftrightarrow\frac{16R^5-16R^4r+8R^3r^2-16R^2r^3+15Rr^4+2r^4}{16R^3r^2}\leq\frac{9R^3}{32r^3}$$

$$\Leftrightarrow 9R^6-32R^5r+32R^4r^2-16R^3r^3+32R^2r^4-30Rr^5-2r^6\geq 0$$

$\Leftrightarrow (R-2r)(9R^5-14R^4r+4R^3r^2-8R^2r^3+16Rr^4+2r^5)\geq 0$, which is true from $R\geq 2r$ $(Euler)$, where it was used,

$$\sum_{cyc}\frac{m_a^2}{a^2}=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(12R^2+8Rr-r^2)-r^3(4R+r)^3}{16R^2r^2s^2}$$

Equality if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$\frac{9}{4}\leq\left(2-\frac{r}{R}\right)^2\leq\sum_{cyc}\frac{m_aw_a}{a^2}\leq\frac{9}{4}\left(\frac{R}{2r}\right)^3$$

Marin Chirciu

Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

Reference:

ROMANIAN MATHEMATICAL MAGAZINE-Interactive Journal-www.ssmrmh.ro