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ABOUT AN INEQUALITY BY LAZAROS ZACHARIADIS-I

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1) If $a,b,c>0$ then:

$$\sum\frac{(b+c)^4}{3a^3+13abc}\ge 3\sqrt[3]{abc}$$

Lazaros Zachariadis-Greece

Solution: Due to homogeneity, we can consider $abc=1$.The problem can be reformulated

Lemma:

2) If $a,b,c>0$ and $abc=1$ then:

$$\sum\frac{(b+c)^4}{3a^3+13}\ge 3$$

Proof: Using Bergstrom's inequality we obtain:

$$LHS=\sum\frac{(b+c)^4}{3a^3+13}\ge\frac{[\sum(b+c)]^2}{\sum(3a^3+13)}=\frac{[\sum(b+c)^2]^2}{3\sum a^3+39}\overset{(1)}{\ge}3=RHS$$

where (1) $\Leftrightarrow\frac{[\sum(b+c)^2]^2}{3\sum a^3+39}\ge 3\Leftrightarrow\frac{[\sum(b+c)]^2}{\sum a^3+13}\ge 9$, which follows from pqr method.

We denote $\sum a=p,\sum bc=q,abc=r=1$

We have $\sum a^2=p^2-2q,\sum a^3=p^3-3pq+3r=p^3-3pq+3,p^2\ge 3q,p\ge 3,q\ge 3$

We obtain:

$$\frac{[\sum(b+c)^2]^2}{\sum a^3+13}=\frac{[\sum(b^2+c^2+2bc)]^2}{\sum a^3+13}=\frac{(2\sum a^2+2\sum bc)^2}{\sum a^3+13}=\frac{4(\sum a^2+\sum bc)^2}{\sum a^3+13}=$$

$$=\frac{4(p^2-2q+q)^2}{(p^3-3pq+3)+13}=\frac{4(p^2-q)^2}{p^3-3pq+16}\overset{(2)}{\ge}9$$

where (2) $\Leftrightarrow\frac{4(p^2-q)^2}{p^3-3pq+16}\ge 9\Leftrightarrow 4p^4-9p^3-8p^2q+27pq+4q^2-144\ge 0$

We consider the function $f:(0,\infty)\to\mathbb{R},f(p)=4p^4-9p^3-8p^2q+27pq+4q^2-144$

Let's prove that $f(p)\ge 0,p>0$.

We calculate $f'(p)=16p^3-27p^2-16pq+27q+4q^2$

It follows: $f'(p)=16p^3-27p^2-16pq+27q+4q^2=16p(p^2-q)-27(p^2-q)+4q^2$

$=(p^2-q)(16p-27)+4q^2>0$, which follows from

$(p^2-q)>0,(16p-27)>0$ and $4q^2>0$

From $f'(p)>0,p>0$ it follows that the function f is strictly increasing on $(0,\infty)$.

As $p\ge 3$ and the function f is strictly increasing on $(0,\infty)$ it follows that: $f(p)\ge f(3)$

Let's calculate $f(3)=4\cdot 3^4-9\cdot 3^3-8\cdot 3^2q+27\cdot 3q+4q^2-144=4q^2+9q-63$

Because $q\ge 3$ we obtain that $f(3)=4q^2+9q-63\ge 4\cdot 3^2+9\cdot 3-63=0$

We've obtained that $f(p) \geq f(3) \geq 0$ and the problem is done. Equality holds if and only if $a = b = c = 1$.

Remark: The problem can be developed.

3) If $a, b, c > 0$ then:

$$\sum \frac{(b+c)^4}{a^3+\lambda abc} \geq \frac{48}{\lambda+1}\sqrt[3]{abc}, \text{ where } 2 \leq \lambda \leq 5$$

Marin Chirciu

Solution: Due to the homogeneity, we can consider $abc = 1$.The problem can be reformulated. **Lemma:**

4) If $a, b, c > 0$ and $abc = 1$ then:

$$\sum \frac{(b+c)^4}{a^3+\lambda} \geq \frac{48}{\lambda+1}, \text{ where } 2 \leq \lambda \leq 5$$

Proof: Using Bergstrom inequality we obtain:

$$LHS = \sum \frac{(b+c)^4}{a^3+\lambda} \geq \frac{[\sum(b+c)^2]^2}{\sum(a^3+\lambda)} = \frac{[\sum(b+c)^2]^2}{\sum a^3 + 3\lambda} \overset{(1)}{\geq} \frac{48}{\lambda+1} = RHS$$

where (1) $\Leftrightarrow \frac{[\sum(b+c)^2]^2}{\sum a^3+3\lambda} \geq \frac{48}{\lambda+1} \Leftrightarrow \frac{[\sum(b+c)^2]^2}{\sum a^3+3\lambda} \geq \frac{48}{\lambda+1}$, which follows from pqr method.

We denote $\sum a = p, \sum bc = q, abc = r = 1$.We have $\sum a^2 = p^2 - 2q$, $\sum a^3 = p^3 - 3pq + 3r = p^3 - 3pq + 3, p^2 \geq 3q, p \geq 3, q \geq 3$. We obtain:

$$\frac{[\sum(b+c)^2]^2}{\sum a^3 + 3\lambda} = \frac{[\sum(b^2+c^2+2bc)]^2}{\sum a^3 + 3\lambda} = \frac{(2\sum a^2 + 2\sum bc)^2}{\sum a^3 + 3\lambda} = \frac{4(\sum a^2 + \sum bc)^2}{\sum a^3 + 3\lambda} =$$

$$= \frac{4(p^2-2q+q)^2}{(p^3-3pq+3)+3\lambda} = \frac{4(p^2-q)^2}{p^3-3pq+3\lambda+3} \overset{(2)}{\geq} \frac{48}{\lambda+1}$$

where (2)$\Leftrightarrow \frac{4(p^2-q)^2}{p^3-3pq+3\lambda+3} \geq \frac{48}{\lambda+1} \Leftrightarrow \frac{(p^2-q)^2}{p^3-3pq+3\lambda+3} \geq \frac{12}{\lambda+1} \Leftrightarrow$

$$\Leftrightarrow (\lambda+1)p^4 - 12p^3 - 2(\lambda+1)p^2q + 36pq + (\lambda+1)q^2 - 36(\lambda+1) \geq 0$$

We consider the function $f: (0, \infty) \to \mathbb{R}$

$$f(p) = (\lambda+1)p^4 - 12p^3 - 2(\lambda+1)p^2q + 36pq + (\lambda+1)q^2 - 36(\lambda+1)$$

Let's prove that: $f(p) \geq 0, p > 0$

We calculate $f'(p) = 4(\lambda+1)p^3 - 36p^2 - 4(\lambda+1)pq + 36q + (\lambda+1)q^2$

It follows: $f'(p) = 4(\lambda+1)p^3 - 36p^2 - 4(\lambda+1)pq + 36q + (\lambda+1)q^2$

$$= 4(\lambda+1)p(p^2-q) - 36(p^2-q) + (\lambda+1)q^2 =$$

$$= 4(p^2-q)\big((\lambda+1)p - 9\big) + (\lambda+1)q^2 > 0$$

which follows from $(p^2-q) > 0, \big((\lambda+1)p - 9\big) \geq 0$, from $\lambda \geq 2$ and $(\lambda+1)q^2 > 0$

From $f'(p) > 0, p > 0$ it follows that the function f is increasing on $(0, \infty)$

As $p \geq 3$ and the function f is strictly increasing on $(0, \infty)$ it follows that: $f(p) \geq f(3)$

Let's calculate

$$f(3) = (\lambda+1)\cdot 3^4 - 12\cdot 3^3 - 2(\lambda+1)\cdot 3^2 q + 36\cdot 3q + (\lambda+1)q^2 - 36(\lambda+1) =$$
$$= (\lambda+1)q^2 + 18(5-\lambda)q + 9(5\lambda - 31)$$

Because $q \geq 3$ and $\lambda \leq 5$ we obtain that:

$$f(3) = (\lambda+1)q^2 + 18(5-\lambda)q + 9(5\lambda-31) \geq (\lambda+1)\cdot 3^2 + 18(5-\lambda)\cdot 3 + 9(5\lambda-31) = 0$$

We've obtained that $f(p) \geq f(3) \geq 0$ and, so, the problem is done.

Equality holds if and only if $a = b = c = 1$.

Note: For $\lambda = \frac{13}{3}$ we obtain the proposed problem by **Lazaros Zachariadis** in RMM 1/2020.

Reference:

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