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ABOUT AN INEQUALITY BY IONUȚ FLORIN VOINEA-I

By Marin Chirciu – Romania

1) If $a, b, c, d > 1$ then:

$$\sum \log_a \frac{b^3 + c^3 + d^3}{b^2 + c^2 + d^2} \geq 4$$

Ionuț Florin Voinea – Romania

Solution: We prove: Lemma:

2) If $x, y, z > 0$ then:

$$\frac{x^3 + y^3 + z^3}{x^2 + y^2 + z^2} \geq \frac{x + y + z}{3}$$

Proof: We have:

$$\begin{aligned} \frac{x^3 + y^3 + z^3}{x^2 + y^2 + z^2} \geq \frac{x + y + z}{3} &\Leftrightarrow 3(x^3 + y^3 + z^3) \geq (x^2 + y^2 + z^2)(x + y + z) \Leftrightarrow \\ &\Leftrightarrow 2(x^3 + y^3 + z^3) \geq \sum xy(x + y), \text{ which follows from summing the inequalities} \\ &x^3 + y^3 \geq xy(x + y) \Leftrightarrow (x + y)(x - y)^2 \geq 0 \text{ and the analogs.} \end{aligned}$$

Equality holds if and only if $x = y = z$. Let's get back to the main problem.

We write the inequality: Using the Lemma we obtain:

$$\begin{aligned} LHS &= \sum \log_a \frac{b^3 + c^3 + d^3}{b^2 + c^2 + d^2} \stackrel{\text{Lemma}}{\geq} \sum \log_a \frac{b + c + d}{3} \stackrel{AGM}{\geq} \sum \log_a \sqrt[3]{bcd} = \\ &= \frac{1}{3} \sum \log_a (bcd) = \frac{1}{3} \sum (\log_a b + \log_b a) \geq \frac{1}{3} \cdot 12 = 4 = RHS \end{aligned}$$

Equality holds if and only if $a = b = c$. **Remark:** The problem can be developed.

3) If $a, b, c, d > 1$ then:

$$\sum \log_a \frac{b^4 + c^4 + d^4}{b^3 + c^3 + d^3} \geq 4$$

Marin Chirciu

Solution: We prove: Lemma:

4) If $x, y, z > 0$ then:

$$\frac{x^4 + y^4 + z^4}{x^3 + y^3 + z^3} \geq \frac{x + y + z}{3}$$

Proof: We have:

$$\frac{x^4 + y^4 + z^4}{x^3 + y^3 + z^3} \geq \frac{x + y + z}{3} \Leftrightarrow 3(x^4 + y^4 + z^4) \geq (x^3 + y^3 + z^3)(x + y + z) \Leftrightarrow$$

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$\Leftrightarrow 2(x^4 + y^4 + z^4) \geq \sum xy(x^2 + y^2)$, which follows from summing the inequalities $x^4 + y^4 \geq xy(x^2 + y^2) \Leftrightarrow (x - y)(x^3 - y^3) \geq 0$, obvious, because the factors $(x - y)$ and $(x^3 - y^3)$ have the same signs and the analogs. Equality holds if and only if $x = y = z$. Let's get back to the main problem. Inequality can be written:

Using the lemma we obtain:

$$\begin{aligned} LHS &= \sum \log_a \frac{b^4 + c^4 + d^4}{b^3 + c^3 + d^3} \stackrel{\text{Lemma}}{\geq} \sum \log_a \frac{b + c + d}{3} \stackrel{AGM}{\geq} \sum \log_a \sqrt[3]{bcd} = \\ &= \frac{1}{3} \sum \log_a (bcd) = \frac{1}{3} \sum (\log_a b + \log_b a) \geq \frac{1}{3} \cdot 12 = 4 = RHS \end{aligned}$$

Equality holds if and only if $a = b = c$. **Remark:** The problem can be developed.

5) If $a, b, c, d > 1$ then:

$$\sum \log_a \frac{b^{n+1} + c^{n+1} + d^{n+1}}{b^n + c^n + d^n} \geq 4, n \in \mathbb{N}$$

Marin Chirciu

Solution: We prove: **Lemma:**

6) If $x, y, z > 0$ then:

$$\frac{x^{n+1} + y^{n+1} + z^{n+1}}{x^n + y^n + z^n} \geq \frac{x + y + z}{3}$$

Proof: We have:

$$\begin{aligned} \frac{x^{n+1} + y^{n+1} + z^{n+1}}{x^n + y^n + z^n} \geq \frac{x + y + z}{3} &\Leftrightarrow 3(x^{n+1} + y^{n+1} + z^{n+1}) \geq \\ &\geq (x^n + y^n + z^n)(x + y + z) \Leftrightarrow 2(x^{n+1} + y^{n+1} + z^{n+1}) \geq \sum (x^n y + x y^n) \end{aligned}$$

With follows from summing the inequalities $x^{n+1} + y^{n+1} \geq x^n y + x y^n \Leftrightarrow (x - y)(x^n - y^n) \geq 0$, obviously, because the factors $(x - y)$ and $(x^n - y^n)$ have the same sign and the analogs. Equality holds if and only $x = y = z$. Let's get back to the main problem. The inequality can be written:

Using the Lemma we obtain:

$$\begin{aligned} LHS &= \sum \log_a \frac{b^{n+1} + c^{n+1} + d^{n+1}}{b^n + c^n + d^n} \stackrel{\text{Lemma}}{\geq} \sum \log_a \frac{b + c + d}{3} \stackrel{AGM}{\geq} \sum \log_a \sqrt[3]{bcd} = \\ &= \frac{1}{3} \sum \log_a (bcd) = \frac{1}{3} \sum (\log_a b + \log_b a) \geq \frac{1}{3} \cdot 12 = 4 = RHS \end{aligned}$$

Equality holds if and only if $a = b = c$.

Note: For $n = 2$ we obtain the proposed problem by Ionuț Florin Voinea in RMM 11/2020.

Reference:

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