SOME LIMITS OF SEQUENCES OF BĂTINEŢU AND LALESCU TYPE

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Abstract. In this paper we present some limits of Bătineţu and Lalescu sequences.

Problem 1.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty}\frac{a_n}{n}=a\in\mathbb{R}_+^* \text{ and } \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=b\in\mathbb{R}_+^*, \text{ then } \lim_{n\to\infty}(a_{n+1}-a_n)=a\ln b$$

Proof. We have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{n+1}\cdot\frac{n}{a_n}\cdot\frac{n+1}{n}=a\cdot\frac{1}{a}\cdot 1=1$$

We denote $u_n=\dfrac{a_{n+1}}{a_n}$ and $a_{n+1}-a_n=a_n\left(\dfrac{a_{n+1}}{a_n}-1\right)=a_n(u_n-1)=a_n\cdot\dfrac{u_n-1}{\ln u_n}\cdot\ln u_n=$

(1) $$=\frac{a_n}{n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n, \forall n\in\mathbb{N}^*$$

(2) We have $\lim\limits_{n\to\infty}u_n=1$, so $\lim\limits_{n\to\infty}\dfrac{u_n-1}{\ln u_n}=1$; $\lim\limits_{n\to N}u_n^n=\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n=b$

From (1) and (2) we obtain $\lim\limits_{n\to\infty}(a_{n+1}-a_n)=a\cdot 1\cdot\ln b=a\ln b$

Example.

$$\text{If } a_n=\sqrt[n]{n!}, \text{ then } \lim_{n\to\infty}\frac{a_n}{n}=\frac{1}{e}, \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n=e \text{ and}$$

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}\ln e=\frac{1}{e}, \text{ i.e. Traian Lalescu limit}$$

□

Problem 2.
If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequence such that

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}=a\in\mathbb{R}_+^* \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in\mathbb{R}_+^*, \text{ then } \lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}\right)=\frac{ae}{2b}$$

Proof. We have

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{b_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{b_n}}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{b_{n+1}}\cdot\frac{b_n}{n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+1}\frac{b_n n}{b_{n+1}}=\frac{e}{b}$$

$$\lim_{n\to\infty}\frac{a_n}{n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{(n+1)^2-n^2}=\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}\cdot\frac{n}{2n+1}=\frac{a}{2}$$

$$\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}=\frac{a_n}{\sqrt[n]{b_n}}(u_n-1)=\frac{a_n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n=\frac{a_n}{n\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=$$

$$=\frac{a_n}{n^2}\cdot\frac{n}{\sqrt[n]{b_n}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n;\text{ where we denote } u_n=\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{\sqrt[n]{b_n}}{a_n}=$$

$$=\frac{a_{n+1}}{(n+1)^2}\cdot\frac{n^2}{a_n}\cdot\frac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\frac{\sqrt[n]{b_n}}{n}\cdot\frac{n+1}{n}$$

Therefore, $\lim\limits_{n\to\infty}u_n=\frac{a}{2}\cdot\frac{2}{a}\cdot\frac{e}{b}\cdot\frac{b}{e}\cdot 1=1$, so $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$,

$$\lim_{n\to N}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)^n\frac{b_n}{b_{n+1}}\sqrt[n+1]{b_{n+1}}=$$

$$=\lim_{n\to\infty}\left(\frac{b_n n}{b_{n+1}}\cdot\frac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\frac{n+1}{n}\cdot\left(\frac{a_{n+1}}{a_n}\right)^n\right)=$$

$$=\frac{1}{b}\cdot\frac{b}{e}\cdot 1\cdot\lim_{n\to\infty}\left(\left(\left(1+\frac{a_{n+1}-a_n}{a_n}\right)^{\frac{a_n}{a_{n+1}-a_n}}\right)^{\frac{a_{n+1}-a_n}{n}\cdot\frac{n^2}{a_n}}\right)=$$

$$=\frac{1}{e}\cdot e^{a\cdot\frac{2}{a}}=e.\text{ Hence, }\lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}\right)=\frac{a}{2}\cdot\frac{e}{b}\cdot 1\cdot\ln e=\frac{ae}{2b}$$

□

Problem 3.

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=0$$

Proof.

It is well-known that $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$

If we denote $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\left(\frac{n}{n+1}\right)$, then $\lim\limits_{n\to\infty}u_n=1$, $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!}{n!}\left(\frac{n}{n+1}\right)^n\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}\cdot e=1$$

Hence,

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}-\frac{\sqrt[n]{n!}}{n}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\right)^2\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n=\frac{1}{e^2}\cdot 1\cdot\ln 1=0$$

□

Probelm 4.

$$\lim_{n\to\infty}\sqrt[n]{(2n-1)!!}\left(\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2}-\frac{\sqrt[n]{n!}}{n^2}\right)=-\frac{4}{e^3}$$

Proof.

If it's well-known that $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$ and respectively $\lim\limits_{n\to\infty}\frac{\sqrt[n]{(2n-1)!}}{n}=\frac{2}{e}$

If we denote $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\left(\frac{n}{n+1}\right)^2$, we denote $\lim\limits_{n\to\infty}u_n=1$, $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{(n+1)!}{n!}\left(\frac{n}{n+1}\right)^{2n}\frac{1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e^2}\lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}}=\frac{1}{e}$$

$$\text{Hence, } \lim_{n\to\infty} \sqrt[n]{(2n-1)!!}\Big(\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^2} - \frac{\sqrt[n]{n!}}{n^2}\Big) =$$

$$= \lim_{n\to\infty}\Big(\frac{\sqrt[n]{(2n-1)!!}}{n}\Big)^2 \frac{\sqrt[n]{n!}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n = \frac{4}{e^2} \cdot \frac{1}{e} \cdot 1 \cdot \ln\Big(\frac{1}{e}\Big) = -\frac{4}{e^3}$$

□

Problem 5.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{n!}} = a \in \mathbb{R}_+^*, \text{ then } \lim_{n\to\infty} \big(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\big) = \frac{a}{e^2}$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^n}\cdot\frac{n^n}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{n!}}\cdot\Big(\frac{n}{n+1}\Big)^{n+1}\cdot\frac{\sqrt[n]{n!}}{n} =$$

$$= \lim_{n\to\infty} \frac{a}{e}\sqrt[n]{\frac{n!}{n^n}} \overset{\text{C-D'A}}{=} \frac{a}{e}\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{a}{e}\lim_{n\to\infty}\Big(\frac{n}{n+1}\Big)^n = \frac{a}{e^2}$$

$$\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \sqrt[n]{a_n}\cdot(u_n - 1) = \frac{\sqrt[n]{a_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \text{ where}$$

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n+1}{n} \cdot \frac{n}{\sqrt[n]{a_n}}, \forall n \geq 2$$

$$\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1, \lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n\sqrt[n]{n!}} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n}{n+1} \cdot \frac{\sqrt[n]{n!}}{n} = a \cdot \frac{e^2}{a} \cdot 1 \cdot \frac{1}{e} = e$$

$$\text{Hence, } \lim_{n\to\infty}\big(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\big) = \frac{a}{e^2}\cdot 1 \cdot \ln e = \frac{a}{e^2}$$

□

Problem 6.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n \sqrt[n]{n!}} = a \in \mathbb{R}_+^*, \text{ then } \lim_{n\to\infty} \frac{1}{n}\big(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\big) = \frac{2a}{e^3}$$

Proof.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n n^2} = \lim_{n\to\infty} \frac{a_{n+1}}{na_n\sqrt[n]{n!}} \cdot \frac{\sqrt[n]{n!}}{n} = a\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{a}{e}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^2} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{2n}}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{2(n+1)}}\cdot\frac{n^{2n}}{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n n^2}\cdot\Big(\frac{n}{n+1}\Big)^{2n+2} =$$

$$= \frac{a}{e}\cdot\frac{1}{e^2} = \frac{a}{e^3}$$

$$\frac{1}{n}\big(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\big) = \frac{\sqrt[n]{a_n}}{n}\cdot(u_n - 1) = \frac{\sqrt[n]{a_n}}{n^2}\cdot\frac{u_n - 1}{\ln u_n}\cdot\ln u_n^n, \text{ where}$$

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{n^2}{\sqrt[n]{a_n}}, \forall n \geq 2$$

$$\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty}\frac{u_n - 1}{\ln u_n} = 1, \lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} \cdot \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^2 = \frac{a}{e} \cdot \frac{e^3}{a} \cdot 1 = e^2$$

$$\text{Hence, } \lim_{n\to\infty} \frac{1}{n}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \frac{a}{e^3} \cdot 1 \cdot \ln e^2 = \frac{2a}{e^3}$$

□

Problem 7.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{n!}} = a \in \mathbb{R}_+^* \text{ and } x_n = a_1 \cdot \sqrt{a_2} \cdot \sqrt[3]{a_3} \cdot \ldots \sqrt[n]{a_n}, \text{ then } \lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n} = \frac{a}{e^3}$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{x_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{x_n}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)^n} \cdot \frac{n^n}{x_n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{1}{e} \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \frac{1}{e} \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{\text{C-D'A}}{=} \frac{1}{e} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^n} \cdot \frac{n^n}{a_n} =$$

$$= \frac{1}{e} \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{n!}} \left(\frac{n}{n+1}\right)^{n+1} \frac{\sqrt[n]{n!}}{n} =$$

$$= \frac{1}{e} \cdot \frac{a}{e} \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} \overset{\text{C-D'A}}{=} \frac{a}{e^2} \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{a}{e^2} \cdot \frac{1}{e} = \frac{a}{e^3}$$

□

Problem 8.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(2n-1)!!}} = a \in \mathbb{R}_+^* \text{ then } \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right) = \frac{2a}{e^2}$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^n} \cdot \frac{n^n}{a_n} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(2n-1)!!}} \cdot \left(\frac{n}{n+1}\right)^{n+1} \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} =$$

$$= \lim_{n\to\infty} \frac{a}{e} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{C-D'A}}{=} \frac{a}{e} \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \frac{a}{e} \lim_{n\to\infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{2a}{e^2}$$

$$\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n} = \sqrt[n]{a_n} \cdot (u_n - 1) = \frac{\sqrt[n]{a_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \text{ where}$$

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\sqrt[n+1]{a_{n+1}}}{n+1} \cdot \frac{n+1}{n} \cdot \frac{n}{\sqrt[n]{a_n}}, \forall n \geq 2$$

$$\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1,$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(2n-1)!!}} \cdot \frac{n+1}{\sqrt[n+1]{a_{n+1}}} \cdot \frac{n}{n+1} \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} =$$

$$= a \cdot \frac{e^2}{2a} \cdot 1 \cdot \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \frac{e^2}{2} \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{C-D'A}}{=}$$

$$= \frac{e^2}{2} \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} =$$

$$= \frac{e^2}{2} \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n = \frac{e^2}{2} \cdot \frac{1}{e} = e.$$

$$\text{Hence, } \lim_{n\to\infty} (\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{2a}{e^2} \cdot 1 \cdot \ln e = \frac{2a}{e^2}$$

□

Problem 9.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n a_n \sqrt[n]{(2n-1)!!}} = a \in \mathbb{R}_+^* \text{ then } \lim_{n\to\infty} \frac{1}{n} (\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{4a}{e^3}$$

Proof.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n n^2} = \lim_{n\to\infty} \frac{a_{n+1}}{n a_n \sqrt[n]{(2n-1)!!}} \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} = a \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} =$$

$$= a \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{C-D'A}}{=} a \lim_{n\to\infty} \frac{(2n+1)!!}{(2n-1)!!} \cdot \frac{n^n}{(n+1)^{n+1}} = a \lim_{n\to\infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{2a}{e}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^2} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^{2n}}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{2(n+1)}} \cdot \frac{n^{2n}}{a_n} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n n^2} \cdot \left(\frac{n}{n+1}\right)^{2n+2} = \frac{2a}{e} \cdot \frac{1}{e^2} = \frac{2a}{e^3}$$

$$\frac{1}{n} (\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{\sqrt[n]{a_n}}{n} \cdot (u_n - 1) = \frac{\sqrt[n]{a_n}}{n^2} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n, \text{ where}$$

$$u_n = \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}} = \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^2} \cdot \frac{(n+1)^2}{n^2} \cdot \frac{n^2}{\sqrt[n]{a_n}}, \forall n \geq 2$$

$$\lim_{n\to\infty} u_n = 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1, \lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} \cdot \frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} \cdot \left(\frac{n}{n+1}\right)^2 = \frac{2a}{e} \cdot \frac{e^3}{2a} \cdot 1 = e^2$$

$$\text{Hence, } \lim_{n\to\infty} \frac{1}{n} (\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}) = \frac{2a}{e^3} \cdot 1 \cdot \ln e^2 = \frac{4a}{e^3}$$

□

Problem 10.
If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(2n-1)!!}} = a \in \mathbb{R}_+^* \text{ then } \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{e^2}{2a}$$

Proof.

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} \overset{\text{C-D'A}}{=} \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^n} \cdot \frac{n^n}{a_n} =$$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(2n-1)!!}} \cdot \left(\frac{n}{n+1}\right)^{n+1} \cdot \frac{\sqrt[n]{(2n-1)!!}}{n} =$$

$$= \lim_{n\to\infty} \frac{a}{e} \sqrt[n]{\frac{(2n-1)!!}{n^n}} \overset{\text{C-D'A}}{=} \frac{a}{e} \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} =$$

$$= \frac{a}{e} \lim_{n\to\infty} \frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2a}{e^2}$$

$$\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{n^2}{\sqrt[n]{a_n}}(u_n - 1) = \frac{n^2}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} = \frac{n}{\sqrt[n]{a_n}} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n$$

$$\text{where } u_n = \left(\frac{n+1}{n}\right)^2 \cdot \frac{\sqrt[n]{a_n}}{\sqrt[n+1]{a_{n+1}}}, \forall n \geq 2; \lim_{n\to\infty} u_n - 1, \lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} \cdot \frac{a_n}{a_{n+1}} \cdot \sqrt[n+1]{a_{n+1}} = e^2 \lim_{n\to\infty} \frac{na_n}{a_{n+1}} \cdot \frac{\sqrt[n+1]{a_{n+1}}}{n} \cdot \frac{n+1}{n} =$$

$$= e^2 \cdot \frac{e}{2a} \cdot \frac{2a}{e^2} \cdot 1 = e$$

$$\text{Hence, } \lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}}\right) = \frac{e^2}{2a} \cdot 1 \cdot \ln e = \frac{e^2}{2a}$$

□

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