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UP.366. If $s_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} - 2\sqrt{n};\ n \geq 1$ find:

$$\Omega = \lim_{n\to\infty}\left(1 + e^{s_{n+1}} - s^{s_n}\right)^{n\sqrt{n}}$$

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Solution 1 by proposers, Solution 2 by Marian Ursărescu-Romania, Solution 3 by Felix Marin-Romania

Solution 1 by proposers

It's known that: $\lim_{n\to\infty} s_n = s \in (-2,-1)$.

It follows that: $\lim_{n\to\infty}\left(e^{s_{n+1}} - e^{s_n}\right) = e^s - e^s = 0$

$$\Omega = \lim_{n\to\infty}\left(\left(1 + e^{s_{n+1}} - s^{s_n}\right)^{\frac{1}{e^{s_{n+1}}e^{s_n}}}\right)^{n\sqrt{n}(e^{s_{n+1}}-e^{s_n})} = e^{\lim_{n\to\infty} n\sqrt{n}(e^{s_{n+1}}-e^{s_n})};\ (1)$$

$$\lim_{n\to\infty} n\sqrt{n}\left(e^{s_{n+1}} - e^{s_n}\right) = \lim_{n\to\infty} e^{s_n}\left(e^{s_{n+1}-s_n} - 1\right)n\sqrt{n} =$$

$$= e^s \cdot \lim_{n\to\infty}\frac{e^{s_{n+1}-s_n} - 1}{s_{n+1} - s_n}\cdot n\sqrt{n}\left(s_{n+1} - s_n\right) =$$

$$= e^s \cdot 1 \cdot \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n}\right) =$$

$$= e^s \cdot \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n+1}-\sqrt{n}\right)\right) = e^s \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+1}+\sqrt{n}}\right) =$$

$$= e^s \lim_{n\to\infty} n\sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}-2\sqrt{n+1}}{\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right)} =$$

$$= -e^s \lim_{n\to\infty} \frac{n}{\sqrt{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)} \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) =$$

$$= -\frac{1}{2} e^s \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = -\frac{1}{4} e^s; (2)$$

From (1),(2) it follows that:

$$\Omega = \lim_{n\to\infty} \left(1 + e^{s_{n+1}} - s^{s_n}\right)^{n\sqrt{n}} = e^{-\frac{e^s}{4}}$$

Solution 2 by Marian Ursărescu-Romania

$$\Omega = \lim_{n\to\infty} \left(\left(1 + e^{s_{n+1}} - s^{s_n}\right)^{\frac{1}{e^{s_{n+1}}}e^{s_n}}\right)^{n\sqrt{n}\left(e^{s_{n+1}}-e^{s_n}\right)} = e^{\lim_{n\to\infty} n\sqrt{n}\left(e^{s_{n+1}}-e^{s_n}\right)}; (1)$$

$$\lim_{n\to\infty} n\sqrt{n}\left(e^{s_{n+1}} - e^{s_n}\right) = \lim_{n\to\infty} e^{s_n}\left(e^{s_{n+1}-s_n} - 1\right) n\sqrt{n} =$$

$$= e^s \cdot \lim_{n\to\infty} \frac{e^{s_{n+1}-s_n} - 1}{s_{n+1}-s_n} \cdot n\sqrt{n}\left(s_{n+1}-s_n\right); (2)$$

$$\lim_{n\to\infty} e^{s_n} = e^s, \textit{where } s - \textit{is Ioachimescu constant}; (3)$$

$$\lim_{n\to\infty} \frac{e^{s_{n+1}-s_n} - 1}{s_{n+1}-s_n} = \log e = 1; (4)$$

$$\lim_{n\to\infty} n\sqrt{n}\left(s_{n+1}-s_n\right) = \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n}\right) =$$

$$= \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n+1}-\sqrt{n}\right)\right) = \lim_{n\to\infty} n\sqrt{n}\left(\frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n+1}+\sqrt{n}}\right) =$$

$$= \lim_{n\to\infty} n\sqrt{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}-2\sqrt{n+1}}{\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right)} =$$

$$= -\lim_{n\to\infty} \frac{n}{\sqrt{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)} \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) =$$

$$= -\frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = -\frac{1}{4}; (5)$$

From (1), (2), (3), (4), (5) it follows that:

$$\Omega=\lim_{n\to\infty}\left(1+e^{s_{n+1}}-s^{s_n}\right)^{n\sqrt{n}}=\frac{1}{\sqrt[4]{e^{e^s}}}$$

Solution 3 by Felix Marin-Romania

$$s_n=1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}-2\sqrt{n}.$$

Indeed, s_n is related to the zeta function. Namely,

$$s_n=\zeta\left(\frac{1}{2}\right)+\underbrace{\frac{1}{2}\int_n^\infty\frac{\{x\}}{x^{\frac{3}{2}}}dx}_{<\frac{1}{\sqrt{n}}}\ \text{such that}\ \lim_{n\to\infty}s_n=\zeta\left(\frac{1}{2}\right)$$

Moreover,

$$1+e^{s_{n+1}}-e^{s_n}=1+e^{s_n}\left(e^{s_{n+1}-s_n}-1\right)\overset{as\ n\to\infty}{\approx}1+e^{\zeta\left(\frac{1}{2}\right)}\left(s_{n+1}-s_n\right)=$$

$$=1+e^{\zeta\left(\frac{1}{2}\right)}\left(\frac{1}{\sqrt{n+1}}-2\sqrt{n+1}+2\sqrt{n}\right)$$

$$\overset{as\ n\to\infty}{\approx}1+e^{\zeta\left(\frac{1}{2}\right)}\left[\frac{1}{n^{\frac{1}{2}}}\left(1-\frac{1}{2n}\right)-2n^{\frac{1}{2}}\left(1+\frac{1}{2n}-\frac{1}{8n^2}\right)+2n^{\frac{1}{2}}\right]=$$

$$=1-\frac{e^{\zeta\left(\frac{1}{2}\right)}}{4}-\frac{1}{n\sqrt{n}}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\left(1+e^{s_{n+1}}-s^{s_n}\right)^{n\sqrt{n}}=\lim_{n\to\infty}\left\{1-\left[exp\frac{\left(\zeta\left(\frac{1}{2}\right)\right)}{4}\right]\cdot\frac{1}{n\sqrt{n}}\right\}^{n\sqrt{n}}=$$

$$=exp\left(-\frac{\left(\zeta\left(\frac{1}{2}\right)\right)}{4}\right)\approx 0.9436$$