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2401. In ΔABC the following relationship holds:

$$\frac{r}{s}+\sum_{cyc}\frac{n_a}{r_a}\geq 8\cdot\sum_{cyc}\frac{h_a-2r}{g_a}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{s}{r}+\sum\frac{n_a}{r_a}\geq 8\sum\frac{h_a-2r}{g_a}\overset{\because\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}}{\Leftrightarrow}\sum\frac{s}{r_a}+\sum\frac{n_a}{r_a}\geq 8\sum\frac{\frac{2rs}{a}-2r}{g_a}$$

$$\Leftrightarrow\sum\frac{s+n_a}{r_a}\overset{(a)}{\geq}\sum\frac{16r(s-a)}{ag_a}$$

$$Now, Stewart's\ theorem\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bccosA-2bc)$$

$$=as^2-4sbcsin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore s^2-n_a^2=2r_ah_a$$

$$\Rightarrow\frac{a(s+n_a)}{16rr_a(s-a)}=\frac{a(s^2-n_a^2)}{16r\left(\frac{rs}{s-a}\right)(s-a)(s-n_a)}$$

$$=\frac{2ar_ah_a}{16sr^2(s-n_a)}=\frac{2a\left(\frac{rs}{s-a}\right)\left(\frac{2rs}{a}\right)}{16sr^2(s-n_a)}=\frac{s}{4(s-a)(s-n_a)}\overset{?}{\geq}\frac{1}{g_a}\Leftrightarrow\frac{sg_a}{4(s-a)}\overset{?}{\geq}s-n_a$$

$$\Leftrightarrow n_a+\frac{sg_a}{4(s-a)}\underset{(i)}{\overset{?}{\geq}}s$$

$$an_a^2.ag_a^2\geq a^2s^2(s-a)^2$$

$$\Leftrightarrow\{b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\}\{b^2(s-b)+c^2(s-c)$$

$$-a(s-b)(s-c)\}\overset{(1)}{\geq}a^2s^2(s-a)^2$$

$$Let\ s-a=x, s-b=y\ and\ s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x\ andc = x+y\ and\ via\ these\ substitutions,$$

$$(1) \Leftrightarrow \{z(z+x)^2+y(x+y)^2-yz(y+z)\}\{y(z+x)^2+z(x+y)^2-yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2+xz^2+y^3+z^3 \geq 2xyz+yz(y+z) \Leftrightarrow x(y-z)^2+(y+z)(y-z)^2 \geq 0 \to true \Rightarrow (1)\ is\ true \Rightarrow n_a g_a \geq s(s-a)$$

$$\Rightarrow n_a+\frac{sg_a}{4(s-a)} \overset{A-G}{\geq} 2\sqrt{\frac{sn_ag_a}{4(s-a)}} \geq 2\sqrt{\frac{s^2(s-a)}{4(s-a)}} = s \Rightarrow (i)\ is\ true \Rightarrow \frac{a(s+n_a)}{16rr_a(s-a)} \geq \frac{1}{g_a} \Rightarrow \frac{s+n_a}{r_a} \geq \frac{16r(s-a)}{ag_a}\ and\ analogs$$

$$\overset{summing\ up}{\Rightarrow} \sum \frac{s+n_a}{r_a} \geq \sum \frac{16r(s-a)}{ag_a} \Rightarrow (a) is\ true \therefore \frac{s}{r}+\sum \frac{n_a}{r_a} \geq 8\sum \frac{h_a-2r}{g_a}\ (QED)$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc} \frac{n_a}{r_a} \overset{n_a \geq m_a}{\geq} \sum_{cyc} \frac{m_a}{r_a} \overset{AM-GM}{\geq} 3 \cdot \sqrt[3]{\frac{m_am_bm_c}{r_ar_br_c}} \overset{m_am_bm_c \geq r_ar_br_c}{\geq} 3$$

$$\frac{r}{s}+\sum_{cyc} \frac{n_a}{r_a} \overset{Mitrinovic}{\geq} 3\sqrt{3}+3; (1)$$

$$h_a-2r>0; (s>a); (and\ analogs) \to \sum_{cyc} \frac{h_a-2r}{g_a} \overset{g_a \geq h_a}{\leq} \sum_{cyc} \frac{h_a-2r}{h_a} =$$

$$= \sum_{cyc} \left(1-\frac{a}{s}\right) = 3-\frac{2s}{s} = 1$$

$$8 \cdot \sum_{cyc} \frac{h_a-2r}{g_a} \leq 8 \leq 3\sqrt{3}+3 \overset{(1)}{\leq} \frac{s}{r}+\sum_{cyc} \frac{n_a}{r_a}$$

Therefore,

$$\frac{r}{s}+\sum_{cyc} \frac{n_a}{r_a} \geq 8 \cdot \sum_{cyc} \frac{h_a-2r}{g_a}$$

2402. In ΔABC the following relationship holds:

$$\sum \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \le \frac{3}{2}\left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Alex Szoros-Romania

$$a \cdot h_a = 2S \Leftrightarrow h_a = \frac{2S}{a}$$

$$\frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} = \frac{\left(\frac{2S}{a}+\frac{2S}{b}\right)\left(\frac{2S}{a}+\frac{2S}{c}\right)}{\frac{2S}{b}\cdot\frac{2S}{c}} = \frac{(a+b)(a+c)}{a^2} =$$

$$= \frac{a^2+(b+c)a+bc}{a^2} = 1+\frac{b+c}{a}+\frac{bc}{a^2} \le 1+\frac{b+c}{a}+\frac{b^2+c^2}{2a^2}$$

$$\Rightarrow \sum \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \le 3+\sum\frac{b+c}{a}+\frac{1}{2}\sum\frac{b^2+c^2}{a^2} =$$

$$= 3+\sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\sum\left[\left(\frac{a}{b}+\frac{b}{a}\right)^2-2\right] \Rightarrow \text{(Băndilă)}$$

$$\Rightarrow \sum \frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \le \sum\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}\sum\left(\frac{a}{b}+\frac{b}{a}\right)^2 \le$$

$$\le \sum\frac{R}{r}+\frac{1}{2}\left(\frac{R}{r}\right)^2 = \frac{3R}{r}+\frac{3}{2}\left(\frac{R}{r}\right)^2 \quad (1)$$

On the other side $3\frac{R}{r}+\frac{3}{2}\left(\frac{R}{r}\right)^2 \le \frac{3}{2}\left(\frac{R}{r}\right)^3 \Leftrightarrow$

$$\Leftrightarrow 1+\frac{1}{2}\left(\frac{R}{r}\right) \le \frac{1}{2}\left(\frac{R}{r}\right)^2 \Leftrightarrow \left(\frac{R}{r}\right)^2-\frac{R}{r}-2 \ge 0$$

$$\Leftrightarrow \left(\frac{R}{r}-2\right)\left(\frac{R}{r}+1\right) \ge 0 \quad \text{true} \quad (2)$$

From (1) and (2) $\Rightarrow \sum\frac{(h_a+h_b)(h_a+h_c)}{h_b h_c} \le \frac{3}{2}\left(\frac{R}{r}\right)^3$

Solution 2 by Marian Ursărescu -Romania

$$(h_a+h_b)(h_a+h_c) = h_a^2+h_ah_b+h_ah_c+h_bh_c \le w_a^2+w_aw_b+w_bw_c+w_aw_c$$

$$\le w_a^2+w_a^2+w_b^2+w_c^2 \quad (1)$$

But $w_a = \sqrt{s(s-a)} \Rightarrow w_a^2+w_b^2+w_c^2 \le s^2 \quad (2)$

From (1)+(2):

$$(h_a+h_b)(h_a+h_c)\le s(s-a)+s^2=s(s-a+s)=s(b+c)\Rightarrow \text{we must show:}$$

$$s\sum\frac{b+c}{h_bh_c}\le\frac{3}{2}\left(\frac{R}{r}\right)^3\quad(3)$$

$$\sum\frac{b+c}{h_bh_c}=\sum\frac{bc(b+c)}{4S^2}=\frac{1}{4s^2r^2}\cdot\sum bc(b+c)\le\frac{1}{4s^2r^2}\sum(b^3+c^3)=$$

$$=\frac{1}{2s^2r^2}\sum a^3=\frac{1}{2s^2r^2}\cdot 2s(s^2-3r^2-6Rr)=\frac{s^2-3r^2-6Rr}{sr^2}\quad(4)$$

From (3)+(4) we must show:

$$\frac{s(s^2-3r^2-6Rr)}{sr^2}\le\frac{3}{2}\left(\frac{R}{r}\right)^3\Leftrightarrow 2r(s^2-3r^2-6Rr)\le 3R^3\quad(5)$$

From Gerretsen: $s^2\le 4R^2+4Rr+3r^2$ **(6)**

From (5)+(6) we must show:

$$2r(4R^2-2Rr)\le 3R^3\Leftrightarrow 4Rr(2R-r)\le 3R^3\Leftrightarrow 8Rr-4r^2\le 3R^3\Leftrightarrow$$

$$3R^3-8Rr+4r^2\ge 0\Leftrightarrow(3R-r)(R-2r)\ge 0 \text{ true (Euler)}$$

2403. In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian, the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\ge 2\sqrt{2}\cdot\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Now, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore s^2 - n_a^2 = 2r_a h_a$$

$$\Rightarrow \frac{a(s+n_a)}{16rr_a(s-a)} = \frac{a(s^2-n_a^2)}{16r\left(\frac{rs}{s-a}\right)(s-a)(s-n_a)}$$

$$= \frac{2ar_a h_a}{16sr^2(s-n_a)} = \frac{2a\left(\frac{rs}{s-a}\right)\left(\frac{2rs}{a}\right)}{16sr^2(s-n_a)} = \frac{s}{4(s-a)(s-n_a)} \overset{?}{\geq} \frac{1}{g_a} \Leftrightarrow \frac{sg_a}{4(s-a)} \overset{?}{\geq} s - n_a$$

$$\Leftrightarrow n_a + \frac{sg_a}{4(s-a)} \underset{(i)}{\overset{?}{\geq}} s$$

$$an_a^2 . ag_a^2 \geq a^2s^2(s-a)^2$$

$$\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\}\{b^2(s-b) + c^2(s-c)$$

$$- a(s-b)(s-c)\} \overset{(1)}{\geq} a^2s^2(s-a)^2$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$ and via these substitutions,

$$(1) \Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\}$$

$$\geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \to \text{true}$$

$$\Rightarrow (1) \text{ is true} \Rightarrow n_a g_a \geq s(s-a)$$

$$\Rightarrow n_a + \frac{sg_a}{4(s-a)} \overset{A-G}{\geq} 2\sqrt{\frac{sn_a g_a}{4(s-a)}} \geq 2\sqrt{\frac{s^2(s-a)}{4(s-a)}} = s \Rightarrow (i) \text{ is true} \Rightarrow \frac{a(s+n_a)}{16rr_a(s-a)}$$

$$\geq \frac{1}{g_a} \Rightarrow \frac{g_a}{r_a} \geq \frac{8(h_a - 2r)}{n_a + s}$$

$$\Rightarrow \sqrt{\frac{g_a}{r_a}} \geq 2\sqrt{2}\sqrt{\frac{h_a - 2r}{n_a + s}} \text{ and analogs} \overset{\text{summing up}}{\Rightarrow} \sum \sqrt{\frac{g_a}{r_a}} \geq 2\sqrt{2}\sum\sqrt{\frac{h_a - 2r}{n_a + s}} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$h_a - 2r = 2r\left(\frac{s}{a} - 1\right) = \frac{2r(s-a)}{a} = \frac{rh_a}{r_a}, n_a \geq h_a; (\text{and analogs}) \to$$

$$\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}\le\sqrt{r}\cdot\sum_{cyc}\sqrt{\frac{h_a}{n_a}}\cdot\frac{1}{\sqrt{h_a+s}}$$

$$a\ge b\ge c\to h_a\le h_b\le h_c \text{ and } r_a\ge r_b\ge r_c\to\sqrt{\frac{h_a}{r_a}}\ge\sqrt{\frac{h_b}{r_b}}\ge\sqrt{\frac{h_c}{r_c}} \text{ and}$$

$$\frac{1}{\sqrt{h_a+s}}\ge\frac{1}{\sqrt{h_b+s}}\ge\frac{1}{\sqrt{h_c+s}}.\text{Using Chebyshev's}\to$$

$$\sum_{cyc}\sqrt{\frac{h_a}{r_a}}\cdot\frac{1}{\sqrt{h_a+s}}\le\frac{1}{3}\left(\sum_{cyc}\sqrt{\frac{h_a}{r_a}}\right)\left(\sum_{cyc}\frac{1}{\sqrt{h_a+s}}\right)\overset{h_a\le g_a}{\le}\frac{1}{3\sqrt{s}}\left(\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\right)\left(\sum_{cyc}\frac{\sqrt{a}}{2r+a}\right)\ge$$

$$\overset{CBS}{\le}\frac{1}{3\sqrt{s}}\left(\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\right)\left(\sqrt{3\sum_{cyc}\frac{a}{2r+a}}\right)$$

$$\text{Let } f(x)=\frac{x}{2r+x};(x>0),\text{we have}: f'(x)=\frac{2r}{(x+2r)^2} \text{ and } f''(x)=-\frac{4r}{(x+2r)^2}<0$$

$$\text{From Jensen}\to\sum_{cyc}\frac{a}{2r+a}=\sum_{cyc}f(a)\le 3f\left(\frac{\sum a}{3}\right)=3\cdot\frac{\frac{2s}{3}}{2r+\frac{2s}{3}}=\frac{3s}{3r+s}$$

$$\to 2\sqrt{2}\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}\le\frac{2\sqrt{2r}}{3\sqrt{s}}\left(\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\right)\sqrt{3\cdot\frac{3s}{3r+s}}=\frac{2\sqrt{2r}}{\sqrt{3r+s}}\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\overset{(?)}{\le}\sum_{cyc}\sqrt{\frac{g_a}{r_a}}$$

$$8r\le 3r+s\Leftrightarrow 5r\le s,\text{true from Mitrinovic } s\ge 3\sqrt{3}r\ge 5r$$

Therefore,

$$\sum_{cyc}\sqrt{\frac{g_a}{r_a}}\ge 2\sqrt{2}\cdot\sum_{cyc}\sqrt{\frac{h_a-2r}{n_a+s}}$$

2404. In *any* Δ ABC *holds*:

$$\sum\sqrt{\frac{m_a w_a}{n_a r_a}}\ge 2\sqrt{2}\sum\sqrt{\frac{h_a-2r}{n_a+s}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\sqrt{\frac{m_aw_a}{n_ar_a}}\ge 2\sqrt{2}\sum\sqrt{\frac{h_a-2r}{n_a+s}}\Leftrightarrow\sum\sqrt{\frac{m_aw_a}{n_ar_a}}\ge 2\sqrt{2}\sum\sqrt{\frac{\frac{2rs}{a}-2r}{n_a+s}}$$

$$\Leftrightarrow\sum\sqrt{\frac{m_aw_a}{n_ar_a}}\overset{(a)}{\ge}\sum\sqrt{\frac{16r(s-a)}{a(n_a+s)}}$$

$$\text{Now, Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+s(2bc\cos A-2bc)$$

$$=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\therefore s^2-n_a^2=2r_ah_a$$

$$\Rightarrow\frac{m_aw_aa(s+n_a)}{16rr_an_a(s-a)}=\frac{m_aw_aa(s^2-n_a^2)}{16rr_a(s-a)n_a(s-n_a)}$$

$$=\frac{m_aw_aa(2r_ah_a)}{16rr_a(s-a)n_a(s-n_a)}\overset{\text{Ioscu}}{\ge}\frac{\left(\frac{b+c}{2}\cos\frac{A}{2}\right)\left(\frac{2bc}{b+c}\cos\frac{A}{2}\right)a\left(\frac{2rs}{a}\right)}{8r(s-a)n_a(s-n_a)}$$

$$=\frac{bc\left(\frac{s(s-a)}{bc}\right)s}{4(s-a)n_a(s-n_a)}=\frac{s^2}{4n_a(s-n_a)}\overset{A-G}{\ge}\frac{s^2}{(n_a+s-n_a)^2}=1$$

$$\Rightarrow\frac{m_aw_a}{n_ar_a}\ge\frac{16r(s-a)}{a(n_a+s)}\Rightarrow\sqrt{\frac{m_aw_a}{n_ar_a}}\ge\sqrt{\frac{16r(s-a)}{a(n_a+s)}}\text{ and analogs}\overset{\text{summing up}}{\Rightarrow}\sum\sqrt{\frac{m_aw_a}{n_ar_a}}$$

$$\ge\sum\sqrt{\frac{16r(s-a)}{a(n_a+s)}}\Rightarrow(a)\text{ is true}$$

$$\therefore\sum\sqrt{\frac{m_aw_a}{n_ar_a}}\ge 2\sqrt{2}\sum\sqrt{\frac{h_a-2r}{n_a+s}}\ \text{(QED)}$$

2405. In ΔABC the following relationship holds:

$$\frac{r_a r_b r_c}{w_a w_b w_c}+\sum\frac{s_a}{s_a+m_a}\ge\frac{5}{2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{r_a r_b r_c}{w_a w_b w_c}+\sum\frac{s_a}{s_a+m_a}\overset{(*)}{\ge}\frac{5}{2}$$

$$We\ have:\ \frac{r_a r_b r_c}{w_a w_b w_c}=s^2 r\prod\frac{b+c}{2bc.\cos\frac{A}{2}}=\frac{s^2 r}{4sRr.\frac{s}{4R}}.\frac{(a+b)(b+c)(c+a)}{8abc}$$

$$=\frac{(a+b)(b+c)(c+a)}{8abc}$$

$$and\ \sum\frac{s_a}{s_a+m_a}=\sum\frac{1}{1+\frac{m_a}{s_a}}=\sum\frac{1}{1+\frac{b^2+c^2}{2bc}}=\sum\frac{2bc}{(b+c)^2}$$

$$\to(*)\leftrightarrow\frac{\prod(a+b)}{8abc}+\sum\frac{2bc}{(b+c)^2}\ge\frac{5}{2}\leftrightarrow\frac{\prod(b+c)-8abc}{8abc}\ge\sum\left(\frac{1}{2}-\frac{2bc}{(b+c)^2}\right)$$

$$\leftrightarrow\frac{\sum a(b-c)^2}{8abc}\ge\sum\frac{(b-c)^2}{2(b+c)^2}\leftrightarrow\sum(b-c)^2\left(\frac{1}{4bc}-\frac{1}{(b+c)^2}\right)\ge 0$$

$$\leftrightarrow\sum\frac{(b-c)^4}{4bc(b+c)^2}\ge 0, which\ is\ true.$$

$$Therefore,\quad \frac{r_a r_b r_c}{w_a w_b w_c}+\sum\frac{s_a}{s_a+m_a}\ge\frac{5}{2}$$

2406. If m is positive real number and n is integer number, ABC is a triangle with F area and

$(F_n)_{n\ge 0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$, then:

$$\frac{a^{2(m+1)}}{(F_n m_a^2+F_{n+1}m_b^2)^m}+\frac{b^{2(m+1)}}{(F_n m_b^2+F_{n+1}m_c^2)^m}+\frac{c^{2(m+1)}}{(F_n m_c^2+F_{n+1}m_a^2)^m}\ge\frac{4^{m+1}}{3^m F_{n+2}^m}F$$

Proposed by D.M.Bătineţu-Giurgiu, Neculai Stanciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc}\frac{a^{2(m+1)}}{\left(F_n m_a^2+F_{n+1}m_b^2\right)^m}=\sum_{cyc}\frac{(a^2)^{m+1}}{\left(F_n m_a^2+F_{n+1}m_b^2\right)^m}\overset{Holder}{\geq}$$

$$\geq\frac{\left(\sum a^2\right)^{m+1}}{\left((F_n+F_{n+1})\sum m_a^2\right)}\overset{F_n+F_{n+1}}{=}\frac{\left(\sum a^2\right)^{m+1}}{F_{n+2}^m\cdot\left(\frac{3}{4}\sum a^2\right)^m}=\frac{\left(\sum a^2\right)^{m+1}}{f_{n+2}^m\cdot\left(\frac{3}{4}\right)^m\left(\sum a^2\right)^m}=$$

$$=\frac{\sum a^2}{F_{n+2}^m\cdot\frac{3^m}{4^m}}\overset{I-W}{\geq}\frac{4^{m+1}}{3^m F_{n+2}^m}F$$

2407. In ΔABC the following relationship holds:

$$\sum\frac{p-a}{r_a-r}\geq 2\sum\frac{m_a}{b+c}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum\frac{m_a}{b+c}\overset{Panaitopol}{\leq}\sum\frac{Rh_a}{2r(b+c)}=Rp\sum\frac{1}{ab+ac}\overset{CBS}{\leq}Rp\sum\frac{1}{4}\left(\frac{1}{ab}+\frac{1}{ac}\right)$$

$$=\frac{Rp}{2}\sum\frac{1}{ab}=\frac{Rp}{2}\cdot\frac{2p}{4Rrp}=\frac{p}{4r}\ \rightarrow\ 2\sum\frac{m_a}{b+c}\leq\frac{p}{2r}\ (1)$$

$$And:\sum\frac{p-a}{r_a-r}=\frac{1}{r}\sum\frac{p-a}{\frac{p}{p-a}-1}=\frac{1}{r}\sum\frac{(p-a)^2}{a}\overset{CBS}{\geq}\frac{1}{r}\cdot\frac{\left[\sum(p-a)\right]^2}{\sum a}$$

$$=\frac{1}{r}\cdot\frac{p^2}{2p}=\frac{p}{2r}\overset{(1)}{\geq}2\sum\frac{m_a}{b+c}$$

$$Therefore,\quad\sum\frac{p-a}{r_a-r}\geq 2\sum\frac{m_a}{b+c}$$

Solution 2 by Marian Ursărescu-Romania

$$\sum_{cyc}\frac{s-a}{r_a-r}=\sum_{cyc}\frac{s-a}{F\left(\frac{1}{s-a}-\frac{1}{s}\right)}=\frac{1}{sr}\sum_{cyc}\frac{s-a}{\frac{a}{s(s-a)}}=\frac{1}{r}\sum_{cyc}\frac{(s-a)^2}{a}=$$

$$=\frac{1}{r}\cdot\frac{s(s^2+r^2-12Rr)}{4Rr}=\frac{s(s^2+r^2-12Rr)}{4Rr^2};(1)$$

$$m_a \le \frac{a^2+b^2+c^2}{2\sqrt{3}a} \to 2\sum_{cyc}\frac{m_a}{b+c} \le \frac{(a^2+b^2+c^2)}{\sqrt{3}}\sum_{cyc}\frac{1}{ab+ac}$$

$$But: \frac{1}{x+y} \le \frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right) \to \sum_{cyc}\frac{1}{ab+ac} = \sum_{cyc}\frac{1}{a(b+c)} \le \frac{1}{4}\sum_{cyc}\left(\frac{1}{ab}+\frac{1}{ac}\right) = \frac{1}{2}\sum_{cyc}\frac{1}{ab}$$

$$\to 2\sum_{cyc}\frac{m_a}{b+c} \le \frac{a^2+b^2+c^2}{2\sqrt{3}}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) = \frac{a^2+b^2+c^2}{4\sqrt{3}Rr}; (2)$$

From $(1),(2)$ *we must show that:*

$$\frac{s(s^2+r^2-12Rr)}{4Rr^2} \ge \frac{a^2+b^2+c^2}{4\sqrt{3}Rr} \Leftrightarrow$$

$$\sqrt{3}s(s^2+r^2-12Rr) \ge r\cdot 2(s^2-r^2-4Rr); (3)$$

$But: s \ge 3\sqrt{3}r; (Mitrinovic)$. *From* $(3),(4)$ *we must show that:*

$$9(s^2+r^2-12Rr) \ge 2(s^2-r^2-4Rr) \Leftrightarrow 7s^2 \ge 100Rr-11r^2; (5)$$

$$But: s^2 \ge 16Rr-5r^2; (6)(Gerretsen)$$

From $(5),(6)$ *we must show that:*

$$112Rr-35r^2 \ge 100Rr-11r^2 \Leftrightarrow 12Rr \ge 24r^2 \Leftrightarrow R \ge 2r\ (Euler)$$

2408. In ΔABC the following relationship holds:

$$12r \le \frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \le 2R\left(\frac{2R}{r}-1\right)$$

Proposed by Kostas Geronikolas-Greece

Solution by Marian Ursărescu-Romania

For LHS: $\frac{a^2}{m_a}+\frac{b^2}{m_b}+\frac{c^2}{m_c} \ge 3\cdot\sqrt[3]{\frac{(abc)^2}{m_am_bm_c}}$. *We must show that:*

$$3\cdot\sqrt[3]{\frac{(abc)^2}{m_am_bm_c}} \ge 12r \Leftrightarrow \frac{(abc)^2}{m_am_bm_c} \ge 64r^3; (1)$$

But $abc = 4Rrs$ *and* $m_am_bm_c \le \frac{Rs^2}{2}; (2)$. *From* $(1),(2)$ *we must to prove that:*

$$\frac{2\cdot 16R^2r^2s^2}{Rs^2} \ge 64r^3 \Leftrightarrow R \ge 2r\ (Euler)$$

For RHS: $m_a \geq \frac{b^2+c^2}{4R} \geq \frac{bc}{2R}$. *So, we must show that:*

$$2R\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right) \leq 2R\left(\frac{2R}{r}-1\right) \Leftrightarrow \frac{a^3+b^3+c^3}{abc} \leq \frac{2R-r}{r} \Leftrightarrow$$

$$\frac{2s(s^2-3r^2-6Rr)}{4Rrs} \leq \frac{2R-r}{r} \Leftrightarrow s^2-3r^2-6Rr \leq 4R^2-2Rr$$

$\Leftrightarrow s^2 \leq 4R^2+4Rr+3r^2$, *which is true from Gerretsen's inequality.*

2409. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}} \geq 1-\frac{r}{2R}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

$$2\frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}}+3\sin^2 \frac{B}{2} = \frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}}+\frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}}+\sin^2 \frac{B}{2}+\sin^2 \frac{B}{2}+\sin^2 \frac{B}{2} \geq$$

$$\geq 5\sqrt[5]{\frac{\sin^{10} \frac{A}{2}}{\sin^6 \frac{A}{2}} \cdot \sin^6 \frac{A}{2}} = 5\sin^2 \frac{A}{2}; (and\ analogs)$$

$$\sum_{cyc} \frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}} \geq \sin^2 \frac{A}{2}+\sin^2 \frac{B}{2}+\sin^2 \frac{C}{2}; (1)$$

$$But: \sin^2 \frac{A}{2}+\sin^2 \frac{B}{2}+\sin^2 \frac{C}{2} = 1-\frac{r}{2R}; (2)$$

From $(1),(2)$ *it follows that:*

$$\sum_{cyc} \frac{\sin^5 \frac{A}{2}}{\sin^3 \frac{B}{2}} \geq 1-\frac{r}{2R}$$

2410. In ΔABC the following relationship holds:

$$a^a \cdot b^b \cdot c^c \cdot \left(6\sqrt{3}r\right)^{6\sqrt{3}r} \leq (a^2+b^2+c^2)^{2s}$$

Proposed by Daniel Sitaru – Romania

Solution by Iulian Cristi and Ruxandra Daniela Tonilă-Romania

Recall Cauchy's generalized inequality:

$$x_1^{p_1}x_2^{p_2}\dots x_n^{p_n}\le\left(\frac{x_1p_1+x_2p_2+\dots+x_np_n}{p_1+p_2+\dots+p_n}\right)^{p_1+p_2+\dots+p_n}$$

Take $x_1=p_1=a;x_2=p_2=b,x_3=p_3=c\Rightarrow$

$$\Rightarrow a^ab^bc^c\le\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}=\left(\frac{a^2+b^2+c^2}{2s}\right)^{2s}$$

Our inequality becomes $a^ab^bc^c\left(6\sqrt{3}r\right)^{6\sqrt{3}r}\le\left(\frac{a^2+b^2+c^2}{2s}\right)^{2s}\left(6\sqrt{3}r\right)^{6\sqrt{3}r}$

We shall prove that:

$$\left(\frac{a^2+b^2+c^2}{2s}\right)^{2s}\left(6\sqrt{3}r\right)^{6\sqrt{3}r}\le(a^2+b^2+c^2)^{2s}$$

Using Mitrinovic $\Rightarrow 3\sqrt{3}r\le S\Rightarrow 6\sqrt{3}r\le 2S\Rightarrow\left(6\sqrt{3}r\right)^{6\sqrt{3}r}\le(2s)^{2s}$

$$\Leftrightarrow\left(\frac{1}{2s}\right)^{2s}\left(6\sqrt{3}r\right)^{6\sqrt{3}r}\le|\cdot(a^2+b^2+c^2)^{2s}$$

$$\left(\frac{a^2+b^2+c^2}{2s}\right)^{2s}\left(6\sqrt{3}r\right)^{6\sqrt{3}r}\le(a^2+b^2+c^2)^{2s}\text{ true}$$

2411. In $\Delta ABC, I_a, I_b, I_c$ –excenters, then:

$$\sum\frac{(AI_a)^{n+1}}{b^n}\ge 2^{n+1}\cdot\sqrt{3^{n+4}}\cdot\frac{r^{n+1}}{R^n}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$Let: IP\perp AC,\angle PAI_a=\frac{A}{2},\frac{r_a}{AI_a}=\sin\frac{A}{2}\to\frac{s\cdot\tan\frac{A}{2}}{AI_a}=\sin\frac{A}{2}\to AI_a=\frac{s}{\cos\frac{A}{2}}$$

$$\sum_{cyc}\frac{(AI_a)^{n+1}}{b^n}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{(\prod AI_a)^{n+1}}{(\prod a)^n}}=3\sqrt[3]{\frac{\left(\frac{s^3}{\prod\cos\frac{A}{2}}\right)^{n+1}}{(4RF)^n}}=3\sqrt[3]{\frac{\left(\frac{s^3}{\prod\sqrt{\frac{s(s-a)}{bc}}}\right)^{n+1}}{(4RF)^n}}=$$

$$= 3\sqrt[3]{\frac{\left(\frac{\frac{s^3}{sF}}{abc}\right)^{n+1}}{(4RF)^n}} = 3\sqrt[3]{\frac{\left(\frac{s^3\cdot 4RF}{sF}\right)^{n+1}}{(4RF)^n}} = 3\sqrt[3]{\frac{(4Rs^2)^{n+1}}{(4RF)^n}}$$

$$Need\ to\ show: 3\sqrt[3]{\frac{(4Rs^2)^{n+1}}{(4RF)^n}} \ge 2^{n+1}\cdot\sqrt{3^{n+4}}\cdot\frac{r^{n+1}}{R^n} \to$$

$$27\cdot\frac{(4Rs^2)^{n+1}}{(4RF)^n} \ge 8^{n+1}\cdot 3^{(n+4)\cdot\frac{3}{2}}\cdot\frac{r^{3n+3}}{R^{3n}} \to$$

$$27\cdot\frac{(4R)^n}{(4R)^n}\cdot\frac{s^{2n}}{F^n}\cdot 4RF^2 \ge 8^{n+1}\cdot 3^{\frac{3(n+4)}{2}}\cdot\frac{r^{3n+3}}{R^{3n}} \to$$

$$27\cdot s^n\cdot 4Rs^2\cdot R^{3n} \ge 8^{n+1}\cdot 3^{\frac{3(n+4)}{2}}\cdot r^{4n+3}; (i)$$

$$\because R \ge 2r\ (Euler) \to R^{3n} \ge 8^n\cdot r^{3n} \to 4R \ge 8r$$

$$\because s > 3\sqrt{3}r\ (Mitrinovic) \to s^2 \ge 27r^2 \to s^n \ge 3^{\frac{3n}{2}}\cdot r^n$$

$$27\cdot s^n\cdot 4Rs^2\cdot R^{3n} \ge 27\cdot 3^{\frac{3n}{2}}\cdot r^n\cdot 8r\cdot 27r^2\cdot 8^n\cdot r^{3n} =$$

$$= 8^{n+1}\cdot 3^{\frac{3n}{2}}\cdot 3^3\cdot 3^3\cdot r^{4n+3} = 8^{n+1}\cdot 3^{\frac{3(n+4)}{2}}\cdot r^{4n+3} \to (i) is\ true.$$

Solution 2 by George Florin Şerban-Romania

$$Lemma: AI_a = \frac{(r_a - r)r_a}{r} = \frac{r_a^2}{r} - r_a$$

$$\sum_{cyc}\frac{(AI_a)^{n+1}}{b^n} \overset{Radon}{\ge} \frac{(\sum AI_a)^{n+1}}{(\sum b)^n} = \frac{\left(\frac{1}{r}\sum r_a^2 - \sum r_a\right)^{n+1}}{2^n s^n} \overset{Mitrinovic}{\ge}$$

$$\ge \frac{\left[\frac{(4R+r)^2 - 2s^2}{r} - (4R+r)\right]^{n+1}}{2^n\left(\frac{3\sqrt{3}R}{2}\right)^n} = \frac{\left[\frac{(4R+r)^2}{r} - \frac{2s^2}{r} - 4R - r\right]^{n+1}}{2^n\cdot\frac{3^{\frac{3n}{2}}R^n}{2^n}} =$$

$$= \frac{(16R^2 + 4Rr - 2s^2)^{n+1}}{3^{\frac{3n}{2}}\cdot r^{n+1}\cdot R^n} = \frac{2^{n+1}(4R^2 - 2Rr - 3r^2)^{n+1}}{3^{\frac{3n}{2}}\cdot r^{n+1}\cdot R^n} \overset{?}{\ge} 2^{n+1}\cdot\sqrt{3^{n+4}}\cdot\frac{r^{n+1}}{R^n}$$

$$\Leftrightarrow (4R^2 - 2Rr - 3r^2)^{n+1} \ge r^{2n+2}\cdot 3^{\frac{3n}{2}}\cdot 3^{\frac{n+4}{2}} = (r^2)^{n+1}\cdot(3^2)^{n+1} = (9r^2)^{n+1}$$

$$4R^2-2Rr-3r^2\ge 9r^2 \Leftrightarrow 4R^2-2Rr-12r^2\ge 0 \Leftrightarrow 4\left(\frac{R}{r}\right)^2-2\left(\frac{R}{r}\right)-12\ge 0$$

$$t=\frac{R}{r}\ge 2 \to 4t^2-2t-12\ge 0 \to (t-2)(4t+6)\ge 0 \ true.$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ AI_a=\frac{r_a}{\sin\frac{A}{2}}=\frac{s\tan\frac{A}{2}}{\sin\frac{A}{2}}=\frac{s}{\cos\frac{A}{2}}\ (and\ analogs)$$

$$Let\ f(x)=\frac{1}{\cos x}, x\in\left(0,\frac{\pi}{2}\right)\to f'(x)=\frac{\sin x}{\cos^2 x}\ and\ f''(x)=\frac{1+\sin^2 x}{\cos^3 x}>0$$

$$\to f-convex.$$

$$\to\sum AI_a=s\sum f\left(\frac{A}{2}\right)\overset{Jensen}{\ge} 3s.f\left(\frac{1}{3}\sum\frac{A}{2}\right)=\frac{3s}{\cos\frac{\pi}{6}}=2\sqrt{3}s\ \to\sum AI_a\ge 2\sqrt{3}s\ (1)$$

$$\sum\frac{(AI_a)^{n+1}}{b^n}\overset{Hölder}{\ge}\frac{(\sum AI_a)^{n+1}}{(\sum b)^n}\overset{(1)}{\ge}\frac{(2\sqrt{3}s)^{n+1}}{(2s)^n}=2\sqrt{3^{n+1}}.s\ \ge$$

$$\overset{Mitrinovic}{\ge} 2\sqrt{3^{n+1}}.3\sqrt{3}r\overset{Euler}{\ge} 2\sqrt{3^{n+4}}.r.\left(\frac{2r}{R}\right)^n$$

$$Therefore,\ \sum\frac{(AI_a)^{n+1}}{b^n}\ge 2^{n+1}\cdot\sqrt{3^{n+4}}\cdot\frac{r^{n+1}}{R^n}$$

2412. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot^5 A}{\cot^3 B}\ge\frac{3(R+r)^2}{s^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\Delta ABC-acute\to\cot A,\cot B,\cot C>0$$

$$\left(\sum_{cyc}\frac{\cot^5 A}{\cot^3 B}\right)\left(\sum_{cyc}\cot B\right)^3\cdot(1+1+1)\overset{Holder}{\ge}\left(\sum_{cyc}\cot A\right)^5$$

$$\rightarrow \sum_{cyc} \frac{\cot^5 A}{\cot^3 B} \geq \frac{1}{3}\left(\sum_{cyc} \cot A\right)^2 = \frac{1}{3}\cdot\left(\frac{s^2 - r^2 - 4Rr}{2sr}\right)^2 \overset{(1)}{\geq} \frac{3(R+r)^2}{s^2}$$

$$(1) \Leftrightarrow \left(\frac{s^2 - r^2 - 4Rr}{2sr}\right)^2 \geq \frac{(3(R+r))^2}{s^2} \Leftrightarrow \frac{s^2 - r^2 - 4Rr}{2sr} \geq \frac{3(R+r)}{s}$$

$$\Leftrightarrow s^2 - r^2 - 4Rr \geq 6r(R+r) \Leftrightarrow s^2 \geq 10Rr + 7r^2$$

But: $s^2 \geq 16Rr - 5r^2$ $(Gerretsen)$. *We need to prove that:*

$16Rr - 5r^2 \geq 10Rr + 7r^2 \Leftrightarrow 6Rr \geq 12r^2 \Leftrightarrow R \geq 2r$ $(Euler) \rightarrow (1)$ *is true. Proved.*

2413. In ΔABC the following relationship holds:

$$\sum \frac{p-a}{h_a - 2r} \leq 2\sum \frac{p-a}{r_a - r}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\ h_a - 2r = 2r\left(\frac{p}{a} - 1\right) = \frac{2r(p-a)}{a} \rightarrow \frac{p-a}{h_a - 2r} = \frac{a}{2r}\ (and\ analogs)$$

$$\rightarrow \sum \frac{p-a}{h_a - 2r} = \sum \frac{a}{2r} = \frac{p}{r}\ \ (1)$$

$$and:\ 2\sum \frac{p-a}{r_a - r} = \frac{2}{r}\sum \frac{p-a}{\frac{p}{p-a} - 1} = \frac{2}{r}\sum \frac{(p-a)^2}{a} \overset{CBS}{\geq} \frac{2}{r}\cdot\frac{[\sum(p-a)]^2}{\sum a}$$

$$= \frac{2}{r}\cdot\frac{p^2}{2p} = \frac{p}{r} \overset{(1)}{=} \sum \frac{p-a}{h_a - 2r}$$

$$Therefore,\quad \sum \frac{p-a}{h_a - 2r} \leq 2\sum \frac{p-a}{r_a - r}$$

2414. In $\Delta ABC, n_a$ –Nagel's cevian, the following relationship holds:

$$\frac{h_a}{n_a} + \frac{h_b}{n_b} + \frac{h_c}{n_c} \geq \frac{3r}{R - r}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let's prove\ that: 2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}=\sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)$$

$$\frac{n_a}{r_a}+\frac{r_a}{n_a}=\frac{n_a^2+r_a^2}{n_ar_a}=\frac{s^2-2r_ah_a+r_a^2}{n_ar_a}$$

$$\frac{s^2-2r_ah_a+r_a^2}{r_a}=\frac{s^2}{r_a}-\frac{4sr}{a}+\frac{sr}{s-a}=\frac{as^2(s-a)-4s^2r^2+\frac{as^2r^2}{s-a}}{a(s-a)r_a}=$$

$$=\frac{as^2(s-a)^2-4s^2r^2(s-a)+s^2r^2a}{asr(s-a)}=$$

$$=s\cdot\frac{a(s-a)^2\cos\frac{A}{2}-4r^2(s-a)\cos\frac{A}{2}+ar^2\cos\frac{A}{2}}{ar(s-a)\cos\frac{A}{2}}=$$

$$=s\cdot\frac{a(s-a)^2\cos\frac{A}{2}-4r^2(s-a)\cos\frac{A}{2}+ar(s-a)\sin\frac{A}{2}}{ar(s-a)\cos\frac{A}{2}}=$$

$$=s\cdot\frac{a(s-a)\cos\frac{A}{2}-4r^2\cos\frac{A}{2}+ar\cdot\sin\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}=$$

$$=s\cdot\frac{16R^2\cdot\cos^3\frac{A}{2}\prod\left(\sin\frac{A}{2}\right)-4r^2\cdot\cos\frac{A}{2}+ar\cdot\sin\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}$$

$$\to\frac{n_a^2+r_a^2}{r_a}=s\cdot\frac{4Rr\cdot\cos^3\frac{A}{2}-4r^2\cdot\cos\frac{A}{2}+4Rr\cdot\cos\frac{A}{2}\sin^2\frac{A}{2}}{ar\cdot\cos\frac{A}{2}}=$$

$$=s\cdot\frac{4R\left(1-\sin^2\frac{A}{2}\right)-4r+4R\cdot\sin^2\frac{A}{2}}{a}=\frac{4s(R-r)}{a}=2\left(\frac{R}{r}-1\right)h_a$$

$$\to\sum_{cyc}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)=\sum_{cyc}\frac{n_a^2+r_a^2}{r_an_a}=2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}$$

$$By\ AM-GM: \frac{n_a}{r_a}+\frac{r_a}{n_a}\ge 2; (and\ analogs)\to 2\left(\frac{R}{r}-1\right)\sum_{cyc}\frac{h_a}{n_a}\ge 6$$

Therefore,

$$\frac{h_a}{n_a}+\frac{h_b}{n_b}+\frac{h_c}{n_c}\geq\frac{3r}{R-r}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$By\ CBS\ inequality, we\ have: \sum_{cyc}\frac{h_a}{n_a}\geq\frac{9}{\sum\frac{n_a}{h_a}}$$

$$\sum_{cyc}\frac{n_a}{h_a}\overset{CBS}{\leq}\sqrt{\left(\sum_{cyc}\frac{n_a^2}{h_a}\right)\left(\sum_{cyc}\frac{1}{h_a}\right)}=\sqrt{\frac{1}{r}\sum_{cyc}\frac{s^2-2r_ah_a}{h_a}}=$$

$$=\sqrt{\frac{s^2}{r^2}-\frac{2}{r}\sum_{cyc}r_a}=\sqrt{\frac{s^2}{r^2}-\frac{2(4R+r)}{r}}\overset{Gerretsen}{\leq}\sqrt{\frac{4R^2+4Rr+3r^2-8Rr-2r^2}{r^2}}=\frac{2R-r}{r}$$

$$\rightarrow\sum_{cyc}\frac{h_a}{n_a}\geq\frac{9r}{2R-r}\overset{(?)}{\geq}\frac{3r}{R-r}$$

$$Therefore,\ \frac{h_a}{n_a}+\frac{h_b}{n_b}+\frac{h_c}{n_c}\geq\frac{3r}{R-r}$$

2415. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_b+h_c}{h_a}\leq\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\frac{s_a}{w_a}\sqrt{\frac{h_a}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{h_b+h_c}{h_a}=\sum_{cyc}h_a\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\sum_{cyc}\frac{b+c}{a};(1)$$

$$s_a=\frac{2bc}{b^2+c^2}\cdot m_a\overset{Tereshin}{\geq}\frac{2bc}{b^2+c^2}\cdot\frac{b^2+c^2}{4R}=\frac{bc}{2R}\rightarrow\frac{s_a}{w_a}\geq\frac{b+c}{4R\cdot\cos\frac{A}{2}};(2)$$

$$\frac{h_a}{r_a}=\frac{2(s-a)}{a}=\frac{2\cdot 4R\cdot\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{4R\cdot\cos\frac{A}{2}\sin\frac{A}{2}}=\frac{2\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}=$$

$$=\frac{2\prod\sin\frac{A}{2}}{\sin^2\frac{A}{2}}=\frac{r}{2R}\cdot\frac{1}{\sin^2\frac{A}{2}};(3)$$

$$\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\frac{s_a}{w_a}\sqrt{\frac{h_a}{r_a}}\overset{(2),(3)}{\geq}\sum_{cyc}\frac{b+c}{4R\cdot\cos\frac{A}{2}\sin\frac{A}{2}}=\sum_{cyc}\frac{b+c}{2R\cdot\sin A}=\sum_{cyc}\frac{b+c}{a}\overset{(1)}{=}\sum_{cyc}\frac{h_b+h_c}{h_a}$$

Therefore,

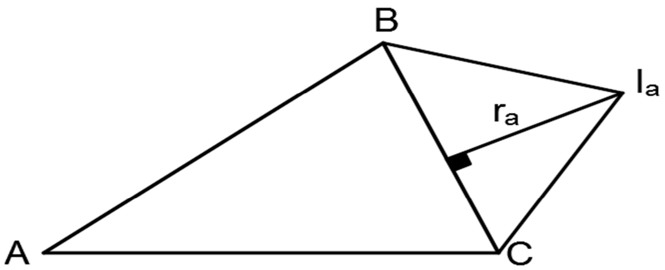
$$\sum_{cyc}\frac{h_b+h_c}{h_a}\leq\sqrt{\frac{2R}{r}}\cdot\sum_{cyc}\frac{s_a}{w_a}\sqrt{\frac{h_a}{r_a}}$$

2416. I_a, I_b, I_c **−excenters in** ΔABC**. Prove that :**

$$\frac{1}{[ABI_c]}+\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}\geq\frac{3}{F}$$

Proposed by Eldeniz Hesenov-Georgia

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ have:\ [BCI_a]=\frac{ar_a}{2}(and\ analogs)$$

$$\to\sum\frac{1}{[BCI_a]}=2\sum\frac{1}{ar_a}\overset{Chebyshev}{\geq}\frac{2}{3}\left(\sum\frac{1}{a}\right)\left(\sum\frac{1}{r_a}\right)\overset{CBS}{\geq}\frac{2}{3}\cdot\frac{9}{\sum a}\cdot\frac{1}{r}=\frac{3}{sr}=\frac{3}{F}$$

$$Therefore,\quad\sum\frac{1}{[BCI_a]}\geq\frac{3}{F}$$

Solution 2 by Avishek Mitra-West Bengal-India

$$PI_a=r_a, BC=a, [BCI_a]=\frac{1}{2}a\cdot r_a$$

$$\sum\frac{1}{[BCI_a]}=\sum\frac{1}{\frac{1}{2}a\cdot r_a}=2\sum\frac{1}{a\cdot\frac{F}{s-a}}=\frac{2}{F}\cdot\sum\frac{s-a}{a}=$$

$$=\frac{2}{F}\left(s\sum\frac{1}{a}-3\right)=\frac{2}{F}\left(s\cdot\frac{\sum ab}{abc}-3\right)=\frac{2}{F}\left(s\cdot\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$

$$=\frac{2}{F}\cdot\frac{s^2-8Rr+r^2}{4Rr}$$

Need to show:

$$\frac{2}{F}\cdot\frac{s^2-8Rr+r^2}{4Rr}\geq\frac{3}{F}\leftrightarrow s^2-8Rr+r^2\geq 6Rr\leftrightarrow s^2\geq 14Rr-r^2$$

But: $s^2\geq 16Rr-5r^2$ $(Gerretsen)$***, so we must to prove thetL***

$$16Rr-5r^2\geq 14Rr-r^2\leftrightarrow 2Rr\geq 4r^2\leftrightarrow R\geq 2r\ (Euler).$$

2417. **If $m\geq 0, u, v>0$ and α,β,γ are the measures of the angles of triangle ABC, then prove that:**

$$\sum_{cyc}\frac{\sin\alpha}{\left(u\sin\beta+v\sqrt{\sin\alpha\sin\beta}\right)^m}\geq\left(\frac{3}{u+v}\right)^m(\sin\alpha+\sin\beta+\sin\gamma)^{1-m}$$

Proposed by D.M.Bătinețu Giurgiu, Neculai Stanciu-Romania

Solution by Tran Hong-DongThap-Vietnam

$$\sum_{cyc}\left(u\sin\beta+v\sqrt{\sin\alpha\sin\beta}\right)=u\sum_{cyc}\sin\alpha+v\sum_{cyc}\sqrt{\sin\alpha\sin\beta}\overset{AM-GM}{\leq}$$

$$\leq u\sum_{cyc}\sin\alpha+v\sum_{cyc}\left(\frac{\sin\alpha+\sin\beta}{2}\right)=(u+v)\sum_{cyc}\sin\alpha\,;(1)$$

$$\left(\sum_{cyc}\frac{\sin\alpha}{\left(u\sin\beta+v\sqrt{\sin\alpha\sin\beta}\right)^m}\right)\left(\sum_{cyc}\left(u\sin\beta+v\sqrt{\sin\alpha\sin\beta}\right)\right)^m(1+1+1)^{1-m-1}\geq$$

$$\overset{Holder}{\geq}\sum_{cyc}\sin\alpha$$

$$\to \sum_{cyc} \frac{\sin\alpha}{\left(u\sin\beta + v\sqrt{\sin\alpha\sin\beta}\right)^m} \ge \frac{\sum\sin\alpha}{3^{-m}\left(\sum\left(u\sin\alpha + v\sqrt{\sin\alpha\sin\beta}\right)\right)^m} \overset{(1)}{\ge}$$

$$\ge 3^m \cdot \frac{\sum\sin\alpha}{\left((u+v)\sum\sin\alpha\right)^m} = 3^m \cdot \frac{\sum\sin\alpha}{(u+v)^m\left(\sum\sin\alpha\right)^m} = \left(\frac{3}{u+v}\right)^m \left(\sum\sin\alpha\right)^{1-m}$$

2418. In $\Delta ABC, g_a$ –Gergonne' s cevian, the following relationship holds :

$$\frac{AH \cdot h_a}{2r} \ge \sqrt{{g_a}^2 + (r_b + r_c - 2r_a)h_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\ \frac{AH.h_a}{2r} \ge \sqrt{{g_a}^2 + (r_b + r_c - 2r_a)h_a}$$

$$We\ know\ that:\ {g_a}^2 = (s-a)^2 + 2rh_a\ and\ r_b + r_c = \frac{2r_br_c}{h_a} = \frac{2s(s-a)}{h_a}$$

$$\to {g_a}^2 + (r_b + r_c - 2r_a)h_a = (s-a)^2 + 2rh_a + \left(\frac{2s(s-a)}{h_a} - 2r_a\right)h_a$$

$$= (s-a)^2 + 2s(s-a) - 2rh_a\left(\frac{s}{s-a} - 1\right) = (s-a)^2 + 2s(s-a) - \frac{2rah_a}{s-a}$$

$$= (s-a)^2 + 2s(s-a) - 4rr_a \overset{rr_a = bc - r_br_c}{\cong} [(s-a)+s]^2 - s^2 - 4bc + 4r_br_c$$

$$= (b+c)^2 - 4bc + 4s(s-a) - s^2 = (b-c)^2 + (b+c)^2 - a^2 - s^2$$

$$\to \sqrt{{g_a}^2 + (r_b + r_c - 2r_a)h_a} = \sqrt{4{m_a}^2 - s^2}$$

$$We\ know\ that:\ AH = 2R|\cos A| \to AH^2 = 4R^2(1 - \sin^2 A) = 4R^2 - a^2$$

$$\to (*) \leftrightarrow \frac{(4R^2 - a^2){h_a}^2}{4r^2} \ge 4{m_a}^2 - s^2 \leftrightarrow 4\left(\frac{Rh_a}{2r}\right)^2 - s^2 \ge 4{m_a}^2 - s^2 \leftrightarrow \frac{Rh_a}{2r} \ge m_a$$

$$which\ is\ Panaitopol\ inequality.$$

$$Therefore,\quad \frac{AH.h_a}{2r} \ge \sqrt{{g_a}^2 + (r_b + r_c - 2r_a)h_a}$$

2419. In ΔABC the following relationship holds:

$$\frac{\left(\frac{\sin A + \sin B}{\sin C}\right)^3 + \left(\frac{\sin B + \sin C}{\sin A}\right)^3 + \left(\frac{\sin C + \sin A}{\sin B}\right)^3 + 3}{\frac{1}{\sin^4 A} + \frac{1}{\sin^4 B} + \frac{1}{\sin^4 C}} \le \frac{81}{16}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

For ΔABC we have: $sinA + sinB + sinC \le \frac{3\sqrt{3}}{2}$, and for $x = sinA, y = sinB, z = sinC$

$$\frac{\left(\frac{sinA+sinB}{sinC}\right)^3+\left(\frac{sinB+sinC}{sinA}\right)^3+\left(\frac{sinC+sinA}{sinB}\right)^3+3}{\frac{1}{sin^4A}+\frac{1}{sin^4B}+\frac{1}{sin^4C}} \le \frac{81}{16}$$

$$\leftrightarrow 16\left(\frac{x+y}{z}\right)^3+16\left(\frac{y+z}{x}\right)^3+16\left(\frac{z+x}{y}\right)^3+48 \le 81\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right)$$

$$\leftrightarrow 16\left(\frac{x^3}{y^3}+\frac{y^3}{z^3}++\frac{z^3}{x^3}+\frac{x^3}{z^3}+\frac{z^3}{y^3}+\frac{y^3}{x^3}\right)+48 \le$$

$$\le \frac{x^2y}{z^3}+\frac{y^2z}{x^3}+\frac{z^2x}{y^3}+\frac{xy^2}{z^3}+\frac{yz^2}{x^3}+\frac{zx^2}{y^3}+48 \le 81\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right) \text{ true, because:}$$

$$9\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right) \ge 16\left(\frac{x^3}{y^3}+\frac{y^3}{z^3}+\frac{z^3}{x^3}\right)$$

$$\leftrightarrow 9((xy)^4+(yz)^4+(zx)^4) \ge 16(x^7yz^4+x^4y^7z+xy^4z^7)$$

$$\leftrightarrow 9((xy)^4+(yz)^4+(zx)^4) \ge 16((xy)^6+(yz)^6+(zx)^6)$$

$$\leftrightarrow 9((xy)^4+(yz)^4+(zx)^4) \ge 16\cdot\frac{9}{16}((xy)^4+(yz)^4+(xz)^4)$$

$$\leftrightarrow 9\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right) \ge 16\left(\frac{x^3}{z^3}+\frac{z^3}{y^3}+\frac{y^3}{x^3}\right)$$

$$\leftrightarrow \frac{1}{x^4}+\frac{1}{y}^4+\frac{1}{z^4} \ge \frac{16}{3} \rightarrow 9\left(\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}\right) \ge 48$$

2420. Let $m, n \in \mathbb{N}, n \ge 1$, In ΔABC the following relationship holds:

$$a)\ \frac{9}{(m+n)(8R^2+4r^2)} \le \sum \frac{1}{ma^2+nb^2} \le \frac{1}{4(m+n)r^2}$$

$$b)\ \frac{2}{(m+n)R} \le \sum \frac{1}{mh_a+nh_b} \le \frac{1}{(m+n)r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a.\,We\ have:\ \sum\frac{1}{ma^2+nb^2}\overset{CBS}{\geq}\frac{9}{\sum(ma^2+nb^2)}=\frac{9}{(m+n)\sum a^2}$$

$$=\frac{9}{2(m+n)(s^2-r^2-4Rr)}\geq$$

$$\overset{Gerretsen}{\geq}\frac{9}{2(m+n)\big((4R^2+4Rr+3r^2)-r^2-4Rr\big)}=\frac{9}{(m+n)(8R^2+4r^2)}$$

$$Now,\ \sum\frac{1}{ma^2+nb^2}\overset{CBS}{\leq}\sum\frac{1}{(m+n)^2}\left(m.\frac{1}{a^2}+n.\frac{1}{b^2}\right)$$

$$=\frac{1}{m+n}\sum\frac{1}{a^2}\overset{Goldstone}{\leq}\frac{1}{m+n}.\frac{1}{4r^2}=\frac{1}{4(m+n)r^2}$$

$$Therefore,\quad \frac{9}{(m+n)(8R^2+4r^2)}\leq\sum\frac{1}{ma^2+nb^2}\leq\frac{1}{4(m+n)r^2}$$

$$b.\,We\ have:\ \sum\frac{1}{mh_a+nh_b}\overset{CBS}{\geq}\frac{9}{\sum(mh_a+nh_b)}$$

$$=\frac{9}{(m+n)\sum h_a}\overset{\sum h_a\leq\sum m_a\leq 4R+r}{\geq}\frac{9}{(m+n)(4R+r)}\geq$$

$$\overset{Euler}{\geq}\frac{9}{(m+n)\left(4R+\frac{R}{2}\right)}=\frac{2}{(m+n)R}$$

$$Now,\quad \sum\frac{1}{mh_a+nh_b}\overset{CBS}{\leq}\sum\frac{1}{(m+n)^2}\left(m.\frac{1}{h_a}+n.\frac{1}{h_b}\right)=\frac{1}{m+n}\sum\frac{1}{h_a}=\frac{1}{(m+n)r}$$

$$Therefore,\quad \frac{2}{(m+n)R}\leq\sum\frac{1}{mh_a+nh_b}\leq\frac{1}{(m+n)r}$$

2421. In $\Delta ABC, n_a$ −Nagel' s cevian, g_a −Gergonne' s cevian, the following relationship holds :

$$\prod\left(\frac{m_a+\sqrt{g_a}\left(\sqrt{n_a}-\sqrt{g_a}\right)}{r_a}+\frac{2h_a}{n_a+s}\right)\leq\frac{s}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{We know that}:\ n_a{}^2+g_a{}^2=4m_a{}^2-2s(s-a)\ \text{and}\ n_ag_a\ge s(s-a)$$

$$\to (n_a+g_a)^2=(n_a{}^2+g_a{}^2)+2n_ag_a\ge[4m_a{}^2-2s(s-a)]+2s(s-a)=(2m_a)^2$$

$$\to n_a+g_a\ge 2m_a\quad (\text{and analogs})$$

$$\frac{m_a+\sqrt{g_a}(\sqrt{n_a}-\sqrt{g_a})}{r_a}$$

$$=\frac{2m_a+2\sqrt{n_ag_a}-2g_a}{2r_a}\overset{AM-GM}{\leq}\frac{2m_a+(n_a+g_a)-2g_a}{2r_a}\overset{(1)}{\leq}\frac{(n_a+g_a)+n_a-g_a}{2r_a}$$

$$\to\frac{m_a+\sqrt{g_a}(\sqrt{n_a}-\sqrt{g_a})}{r_a}\le\frac{n_a}{r_a}$$

$$\to\frac{m_a+\sqrt{g_a}(\sqrt{n_a}-\sqrt{g_a})}{r_a}+\frac{2h_a}{n_a+s}$$

$$\le\frac{n_a}{r_a}+\frac{2h_a(s-n_a)}{s^2-n_a{}^2}\overset{n_a{}^2=s^2-2r_ah_a}{=}\frac{n_a}{r_a}+\frac{2h_a(s-n_a)}{2r_ah_a}=\frac{n_a}{r_a}+\frac{s-n_a}{r_a}=\frac{s}{r_a}$$

$$\to\prod\left(\frac{m_a+\sqrt{g_a}(\sqrt{n_a}-\sqrt{g_a})}{r_a}+\frac{2h_a}{n_a+s}\right)\le\prod\frac{s}{r_a}=\frac{s^3}{rs^2}=\frac{s}{r}$$

$$\text{Therefore,}\quad\prod\left(\frac{m_a+\sqrt{g_a}(\sqrt{n_a}-\sqrt{g_a})}{r_a}+\frac{2h_a}{n_a+s}\right)\le\frac{s}{r}$$

2422. In ΔABC the following relationship holds:

$$a)\quad\sum s_a{}^2+\frac{r^2(R-2r)}{R+r}\overset{(1)}{\leq}\frac{3(s^2-4Rr-r^2)}{2}.$$

$$b)\quad\sum w_a+\sum\frac{g_a|b-c|}{h_a}\ge\sum m_a$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a.\qquad (1)\leftrightarrow\sum s_a{}^2+\frac{r^2(R-2r)}{R+r}\le\sum m_a{}^2$$

$$We\ have:\ m_a{}^2-s_a{}^2=m_a{}^2\left[1-\left(\frac{2bc}{b^2+c^2}\right)^2\right]$$

$$=m_a{}^2.\frac{(b^2-c^2)^2}{(b^2+c^2)^2}\overset{Tereshin}{\geq}\left(\frac{b^2+c^2}{4R}\right)^2.\frac{(b^2-c^2)^2}{(b^2+c^2)^2}$$

$$\to m_a{}^2-s_a{}^2\geq\frac{(b^2-c^2)^2}{16R^2}\ \ (and\ analogs)$$

$$\to\sum(m_a{}^2-s_a{}^2)\geq\frac{1}{16R^2}\sum(b^2-c^2)^2=\frac{1}{8R^2}\left(\sum a^4-\sum(ab)^2\right)=$$

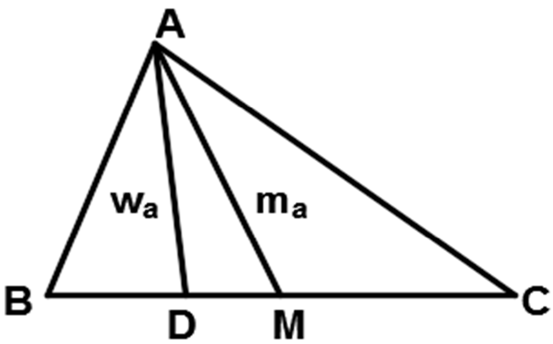
$$=\frac{1}{8R^2}\left(\sum(ab)^2-\left(2\sum(ab)^2-\sum a^4\right)\right)\overset{\sum x^2\geq\sum xy}{\geq}\frac{1}{8R^2}\left(\sum ab.ac-16F^2\right)$$

$$=\frac{4sRr.2s-16s^2r^2}{8R^2}=$$

$$=\frac{2s^2r(R-2r)}{2R^2}\overset{2s^2\geq 27Rr}{\geq}\frac{27r^2(R-2r)}{2R}\geq\frac{27r^2(R-2r)}{2(R+r)}\geq\frac{r^2(R-2r)}{R+r}$$

$$Therefore,\sum s_a{}^2+\frac{r^2(R-2r)}{R+r}\leq\frac{3(s^2-4Rr-r^2)}{2}$$

b.



$$We\ know\ that:DB=\frac{ac}{b+c}\to DM=|BM-DB|=\left|\frac{a}{2}-\frac{ac}{b+c}\right|=\frac{a|b-c|}{2(b+c)}$$

$$w_a+DM\overset{\Delta}{\geq}m_a\leftrightarrow w_a+\frac{a|b-c|}{2(b+c)}\geq m_a$$

$$We\ know\ that:\frac{g_a}{h_a}\geq 1\overset{\Delta}{\geq}\frac{a}{b+c}\geq\frac{a}{2(b+c)}\to w_a+\frac{g_a|b-c|}{h_a}\geq w_a+\frac{a|b-c|}{2(b+c)}\geq m_a$$

$$\rightarrow w_a+\frac{g_a|b-c|}{h_a}\ge m_a\ (and\ analogs)$$

$$Therefore,\quad \sum w_a+\sum\frac{g_a|b-c|}{h_a}\ge\sum m_a$$

2423. In $\Delta ABC, n_a$ −Nagel's cevian, g_a −Gergonne's cevian, the following relationship holds :

$$\sqrt[3]{\frac{\prod(m_aw_a)^2}{\prod({g_a}^2r_ah_a)}}\le 2\left(\frac{2R}{3r}-\frac{5}{6}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ us\ prove:\ m_aw_a\le n_ag_a.$$

$$We\ known\ that:\ {n_a}^2=s(s-a)+\frac{(b-c)^2}{a}s,\qquad {g_a}^2=s(s-a)-\frac{(b-c)^2}{a}(s-a)$$

$$4[(n_ag_a)^2-(m_aw_a)^2]=$$

$$=s(s-a)\left[2(s-a)+\frac{2(b-c)^2}{a}\right]\left[2s-\frac{2(b-c)^2}{a}\right]$$

$$-4\frac{(b+c)^2+(b-c)^2-a^2}{4}\cdot\frac{4bcs(s-a)}{(b+c)^2}$$

$$=s(s-a)\left[(b+c)^2-a^2+4(b-c)^2-\frac{4(b-c)^4}{a^2}-4bc+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$=s(s-a)\left[(b+c)^2-4bc-a^2+\frac{4(b-c)^2(a^2-(b-c)^2)}{a^2}+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$=s(s-a)\left[(b-c)^2-a^2+\frac{4(b-c)^2(a^2-(b-c)^2)}{a^2}+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$=s(s-a)[a^2-(b-c)^2]\left(-1+\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)=4F^2\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right)$$

$$=4F^2(b-c)^2\frac{(2b+2c+a)(2b+2c-a)}{a^2(b+c)^2}\ge 0\rightarrow m_aw_a\le n_ag_a\ (and\ analogs)$$

$$\rightarrow \sqrt[3]{\frac{\prod(m_a w_a)^2}{\prod(g_a{}^2 r_a h_a)}} \le \sqrt[3]{\prod \frac{n_a{}^2}{r_a h_a}} \overset{AM-GM}{\le} \frac{1}{3}\sum \frac{n_a{}^2}{r_a h_a} \overset{n_a{}^2 = s^2 - 2r_a h_a}{\Longleftrightarrow} \frac{1}{3}\sum \frac{s^2 - 2r_a h_a}{r_a h_a}$$

$$= \frac{1}{3}\left(\sum \frac{s^2}{r_a h_a} - \sum 2\right) = \frac{1}{3}\sum \frac{a(s-a)}{2} - 2 = \frac{1}{6r^2}\left(2s^2 - 2(s^2 - r^2 - 4Rr)\right) - 2$$

$$= \frac{r + 4R}{3r} - 2$$

$$Therefore, \quad \sqrt[3]{\frac{\prod(m_a w_a)^2}{\prod(g_a{}^2 r_a h_a)}} \le 2\left(\frac{2R}{3r} - \frac{5}{6}\right)$$

2424. In ΔABC the following relationship holds:

$$12r \le \sum \frac{a^2}{\sqrt{m_b m_c}} \le 3R\sqrt{\frac{2R}{r}}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{a^2}{\sqrt{m_b m_c}} \ge 3\sqrt[3]{\frac{(abc)^2}{m_a m_b m_c}}$$

We must show that:

$$3\sqrt[3]{\frac{(abc)^2}{m_a m_b m_c}} \ge 12r \Leftrightarrow \frac{(abc)^2}{m_a m_b m_c} \ge 64r^3; (1)$$

But: $abc = 4Rrs$ **and** $m_a m_b m_c \le \frac{Rs^2}{2}; (2)$

From $(1), (2)$ **we must show that:**

$$2 \cdot \frac{16R^2 r^2 s^2}{Rs^2} \ge 64r^3 \Leftrightarrow R \ge 2r\ (Euler)$$

$$m_a \ge \frac{b^2 + c^2}{4R} \ge \frac{2bc}{4R} = \frac{bc}{2r} \rightarrow m_b m_c \ge \frac{a^2 bc}{4R^2} \rightarrow \sqrt{m_b m_c} \ge \frac{a\sqrt{bc}}{2R}$$

We must show that:

$$2R\sum_{cyc}\frac{a^2}{a\sqrt{bc}}\le 3R\sqrt{\frac{2R}{r}}\Leftrightarrow\sum_{cyc}\frac{a}{\sqrt{bc}}\le\frac{3}{2}\sqrt{\frac{2R}{r}}\Leftrightarrow\left(\sum_{cyc}\frac{a}{\sqrt{bc}}\right)^2\le\frac{9}{2}\cdot\frac{R}{r};(3)$$

$$\left(\sum_{cyc}\frac{a}{\sqrt{bc}}\right)^2\overset{CBS}{\le}\sum_{cyc}a^2\cdot\sum_{cyc}\frac{1}{bc};(4)$$

From $(3),(4)$ we must show that:

$$\sum_{cyc}a^2\cdot\sum_{cyc}\frac{1}{bc}\le\frac{9}{2}\cdot\frac{R}{r}\Leftrightarrow(a^2+b^2+c^2)\cdot\frac{2s}{abc}\le\frac{9}{2}\cdot\frac{R}{r}$$

$$\Leftrightarrow(a^2+b^2+c^2)\cdot\frac{2s}{4Rrs}\le\frac{9}{2}\cdot\frac{R}{r}\Leftrightarrow a^2+b^2+c^2\le 9R^2\text{ true.}$$

Solution 2 by Ertan Yildirim-Izmir-Turkiye

$$\because m_a\ge h_a;a^2+b^2+c^2=9R^2\ (Leibniz)$$

$$\because\frac{m_a}{h_a}\le\frac{R}{2r}\ (Panaitopol)$$

$$\sum_{cyc}\frac{a^2}{\sqrt{m_bm_c}}\le\sum_{cyc}\frac{a^2}{\sqrt{h_bh_c}}=\sum_{cyc}\frac{a^2}{\sqrt{\frac{ac}{2R}\cdot\frac{ab}{2R}}}=2R\sum_{cyc}\frac{a}{\sqrt{bc}}$$

$$\rightarrow 2R\sum_{cyc}\frac{a}{\sqrt{bc}}\overset{CBS}{\le}2R\sqrt{a^2+b^2+c^2}\cdot\sqrt{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}}\overset{Leibniz}{\le}2R\cdot 3R\sqrt{\frac{a+b+c}{abc}}=$$

$$=2R\cdot 3R\sqrt{\frac{2s}{4Rrs}}=2R\cdot 3r\sqrt{\frac{1}{2Rr}}=3R\sqrt{\frac{2R}{r}}$$

$$\frac{m_a}{h_a}\le\frac{R}{2r}\rightarrow m_a\le\frac{R}{2r}h_a=\frac{R}{2r}\cdot\frac{bc}{2R}=\frac{bc}{4r}$$

$$\sum_{cyc}\frac{a^2}{\sqrt{m_bm_c}}\ge\sum_{cyc}a^2\cdot\frac{4r}{\sqrt{ac\cdot ab}}=4r\sum_{cyc}\frac{a}{\sqrt{bc}}\rightarrow$$

$$8r\sum_{cyc}\frac{a}{2\sqrt{bc}}\overset{AM-GM}{\ge}\sum_{cyc}a^2\cdot\frac{4r}{\sqrt{ab\cdot ac}}=8r\sum_{cyc}\frac{a^2}{a(b+c)}\overset{Bergstrom}{\ge}8r\cdot\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

$$\rightarrow 4r\cdot\frac{(2s)^2}{ab+bc+ca}=4r\cdot\frac{4s^2}{s^2+r^2+4Rr}\overset{?}{\ge}12r$$

$$\rightarrow \frac{4s^2}{s^2+r^2+4Rr} \geq 3 \rightarrow 4s^2 \geq 3s^2+3r^2+12Rr \rightarrow s^2 \geq 3r^2+12Rr$$

But: $s^2 \geq 16Rr-5r^2 \rightarrow 16Rr-5r^2 \geq 3r^2+12Rr \Leftrightarrow R \geq 2r \ (Euler)$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{a^2}{\sqrt{m_b m_c}} \overset{CBS}{\geq} \frac{(\sum a)^2}{\sum \sqrt{m_b m_c}} \overset{\sum xy \leq \sum x^2}{\geq} \frac{4s^2}{\sum m_a} \overset{Leuenberger}{\geq} \frac{4s^2}{4R+r} \overset{Doucet}{\geq} \frac{4.3r(4R+r)}{4R+r} = 12r$$

$$\sum \frac{a^2}{\sqrt{m_b m_c}} = \frac{1}{\sqrt{m_a m_b m_c}} \sum a^2 \sqrt{m_a} \overset{\substack{m_a m_b m_c \geq s^2 r \\ Panaitopol}}{\leq} \frac{1}{\sqrt{s^2 r}} \sum a^2 \sqrt{\frac{Rs}{a}} = \sqrt{\frac{R}{sr}} \sum a\sqrt{a} \leq$$

$$\overset{CBS}{\leq} \sqrt{\frac{R}{sr}} \sqrt{\left(\sum a^2\right)\left(\sum a\right)} \overset{Leibniz}{\leq} \sqrt{\frac{R}{sr}} \sqrt{(9R^2)(2s)} = 3R\sqrt{\frac{2R}{r}}$$

$$Therefore, \quad 12r \leq \sum \frac{a^2}{\sqrt{m_b m_c}} \leq 3R\sqrt{\frac{2R}{r}}$$

2425. In ΔABC the following relationship holds:

$$\frac{4\sqrt{3}}{3R} \leq \frac{cscA}{m_a} + \frac{cscB}{m_b} + \frac{cscC}{m_c} \leq \frac{\sqrt{3}R}{3r^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum \frac{cscA}{m_a} \overset{m_a \geq h_a}{\leq} \sum \frac{cscA}{h_a} = \sum \frac{2R}{a} \cdot \frac{a}{2sr} = \frac{3R}{sr} \overset{Mitrinovic}{\leq} \frac{3R}{3\sqrt{3}r.r} = \frac{\sqrt{3}R}{3r^2}$$

$$\sum \frac{cscA}{m_a} \overset{Panaitopol}{\geq} \sum \frac{cscA}{\frac{Rs}{a}} = \sum \frac{2R}{a} \cdot \frac{a}{Rs} = \frac{6}{s} \overset{Mitrinovic}{\geq} \frac{6.2}{3\sqrt{3}R} = \frac{4\sqrt{3}}{3R}$$

$$Therefore, \quad \frac{4\sqrt{3}}{3R} \leq \sum \frac{cscA}{m_a} \leq \frac{\sqrt{3}R}{3r^2}$$

Solution 2 by Eldeniz Hesenov-Georgia

$$LHS: 2R \sum_{cyc} \frac{1}{am_a} \overset{Chebyshev}{\geq} \frac{2R}{3} \sum_{cyc} \frac{1}{a} \sum_{cyc} \frac{1}{m_a} \overset{Bergstrom}{\geq} \frac{2R}{3} \cdot \frac{9}{s} \cdot \frac{9}{\sum m_a} \geq$$

$$\geq \frac{2R}{3}\cdot\frac{9}{s}\cdot\frac{2}{R} \overset{Mitrinovic}{\geq} \frac{4\sqrt{3}}{3R};(1)$$

$$RHS: 2R\sum_{cyc}\frac{1}{am_a} \overset{m_a\leq h_a}{\leq} 2R\sum_{cyc}\frac{1}{a}\cdot\frac{a}{2F} = \frac{3R}{F} \overset{Mitrinovic}{\leq} \frac{\sqrt{3}R}{3r^2};(2)$$

From $(1),(2)$ it follows that:

$$\frac{4\sqrt{3}}{3R} \leq \frac{cscA}{m_a}+\frac{cscB}{m_b}+\frac{cscC}{m_c} \leq \frac{\sqrt{3}R}{3r^2}$$

2426. In ΔABC, the following relationship holds:

$$\sum\frac{s-a}{h_a-2r} \geq 4\sum\frac{m_a}{b+c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohammed Amine Ben Ajiba-Tanger-Morocco

$$h_a-2r = 2r\left(\frac{s}{a}-1\right) = \frac{2r(s-a)}{a} \to \frac{s-a}{h_a-2r} = \frac{a}{2r} \ (and\ analogs)$$

$$\to \sum\frac{s-a}{h_a-2r} = \sum\frac{a}{2r} = \frac{s}{r} \quad (1)$$

$$4\sum\frac{m_a}{b+c} \overset{Panaitopol}{\leq} 4\sum\frac{Rs}{a(b+c)} = Rs\sum\frac{4}{ab+ac} \overset{CBS}{\leq} Rs\sum\left(\frac{1}{ab}+\frac{1}{ac}\right) = 2Rs\sum\frac{1}{ab}$$

$$= 2Rs.\frac{2s}{4Rsr} = \frac{s}{r} \overset{(1)}{=} \sum\frac{s-a}{h_a-2r}$$

$$Therefore, \quad \sum\frac{s-a}{h_a-2r} \geq 4\sum\frac{m_a}{b+c}$$

2427. In ΔABC the following relationship holds:

$$\frac{3}{4(2R^2+r^2)} \leq \sum\frac{1}{a^2+2b^2} \leq \frac{1}{12r^2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum\frac{1}{a^2+2b^2} \overset{CBS}{\geq} \frac{9}{\sum(a^2+2b^2)} = \frac{3}{\sum a^2} = \frac{3}{2(s^2-r^2-4Rr)} \geq$$

$$\overset{gerretsen}{\geq} \frac{3}{2}\cdot\frac{1}{(4R^2+4Rr+3r^2)-r^2-4Rr}=\frac{3}{4(2R^2+r^2)}$$

$$And:\ \sum\frac{1}{a^2+2b^2}\overset{CBS}{\leq}\sum\frac{1}{9}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{b^2}\right)=\frac{1}{3}\sum\frac{1}{a^2}\overset{Goldstone}{\leq}\frac{1}{3}\cdot\frac{1}{4r^2}=\frac{1}{12r^2}$$

$$Therefore,\quad \frac{3}{4(2R^2+r^2)}\leq\sum\frac{1}{a^2+2b^2}\leq\frac{1}{12r^2}$$

2428. In acute ΔABC holds:

$$\sum\sqrt[5]{\frac{\cos A\sin^4 A}{9}}\leq\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\Delta ABC\ is\ an\ acute\ triangle\rightarrow\cos A,\cos B,\cos C>0$$

$$\sum\sqrt[5]{\frac{\cos A\sin^4 A}{9}}=\sum\sqrt[5]{\cos A\left(\frac{\sin A}{\sqrt{3}}\right)^4}\overset{AM-GM}{\leq}\sum\frac{1}{5}\left(\cos A+4.\frac{\sin A}{\sqrt{3}}\right)=$$

$$=\frac{1}{5}\sum\cos A+\frac{4}{5\sqrt{3}}\sum\sin A=\frac{1}{5}\left(1+\frac{r}{R}\right)+\frac{4}{5\sqrt{3}}\cdot\frac{s}{R}\overset{\substack{Euler\\Mitrinovic}}{\leq}\frac{1}{5}\left(1+\frac{1}{2}\right)+\frac{4}{5\sqrt{3}}\cdot\frac{3\sqrt{3}}{2}=\frac{3}{2}.$$

$$Therefore,\quad \sum\sqrt[5]{\frac{\cos A\sin^4 A}{9}}\leq\frac{3}{2}$$

2429. In $\Delta ABC, n_a$ –Nagel's cevian, $x,y,z>0$, the following relationship holds :

$$\sum\frac{n_a^2}{h_a^2}.x+\frac{1}{r}\sum x.r_a\geq\sum x+\frac{s}{r}\sqrt{\sum xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ known\ that:\ {n_a}^2=s^2-2r_ah_a\ (and\ analogs)$$

$$\rightarrow\sum\frac{{n_a}^2}{{h_a}^2}.x=\sum\frac{s^2-2r_ah_a}{{h_a}^2}.x=\sum\frac{s^2}{{h_a}^2}.x-\sum\frac{2r_a}{h_a}.x=\frac{1}{4r^2}\sum xa^2-\sum\frac{a}{s-a}.x$$

$$\overset{Oppenheim}{\geq}\frac{1}{4r^2}.4F\sqrt{\sum xy}-\sum\frac{s-(s-a)}{s-a}.x=\frac{s}{r}\sqrt{\sum xy}-\frac{1}{r}\sum\frac{sr}{s-a}.x+\sum x$$

$$Therefore,\quad \sum\frac{{n_a}^2}{{h_a}^2}.x+\frac{1}{r}\sum x.r_a\geq\sum x+\frac{s}{r}\sqrt{\sum xy}$$

2430. In $\Delta ABC, n_a$ –Nagel's cevian, $x, y, z > 0$, the following relationship holds :

$$\sum\frac{n_a^2}{h_a^2}.x+\frac{1}{r}\sum x.r_a\geq\sum x+\frac{s}{r}\sqrt{\sum xy}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ known\ that:\ {n_a}^2=s^2-2r_ah_a\ (and\ analogs)$$

$$\rightarrow\sum\frac{{n_a}^2}{{h_a}^2}.x=\sum\frac{s^2-2r_ah_a}{{h_a}^2}.x=\sum\frac{s^2}{{h_a}^2}.x-\sum\frac{2r_a}{h_a}.x=\frac{1}{4r^2}\sum xa^2-\sum\frac{a}{s-a}.x$$

$$\overset{Oppenheim}{\geq}\frac{1}{4r^2}.4F\sqrt{\sum xy}-\sum\frac{s-(s-a)}{s-a}.x=\frac{s}{r}\sqrt{\sum xy}-\frac{1}{r}\sum\frac{sr}{s-a}.x+\sum x$$

$$Therefore,\quad \sum\frac{{n_a}^2}{{h_a}^2}.x+\frac{1}{r}\sum x.r_a\geq\sum x+\frac{s}{r}\sqrt{\sum xy}$$

2431. In $\Delta ABC, I$ –incenter, the following relationship holds :

$$\frac{1}{r}\cdot\sum_{cyc}AI\geq\sum\sqrt{\frac{2(n_a+h_a)}{r_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{AI}{r}\geq\sqrt{\frac{2(n_a+h_a)}{r_a}}\leftrightarrow\frac{1}{\sin^2\frac{A}{2}}\geq\frac{2(n_a+h_a)}{r_a}\leftrightarrow\frac{s\tan\frac{A}{2}}{2\sin^2\frac{A}{2}}-h_a\geq n_a\leftrightarrow$$

$$\left(\frac{s}{\sin A}-\frac{2sr}{a}\right)^2\geq{n_a}^2=s^2-2h_ar_a$$

$$\leftrightarrow \frac{4(R-r)^2s^2}{a^2} \ge s^2\left(1-2\left(\frac{2r}{a}\right)\left(\tan\frac{A}{2}\right)\right) \leftrightarrow 4(R-r)^2$$

$$\ge (2R\sin A)^2 - 4R\sin\frac{A}{2}\cos\frac{A}{2}.4r\tan\frac{A}{2}$$

$$\leftrightarrow (R-r)^2 \ge R^2(1-\cos^2 A) - 2Rr(1-\cos A) \leftrightarrow r^2 \ge -R^2\cos^2 A + 2Rr\cos A$$

$$\leftrightarrow (R\cos A - r)^2 \ge 0, \textit{which is true}.$$

$$\rightarrow \frac{AI}{r} \ge \sqrt{\frac{2(n_a+h_a)}{r_a}} \quad (\textit{And analogs})$$

$$\textit{Therefore,} \quad \frac{1}{r}\sum AI \ge \sum\sqrt{\frac{2(n_a+h_a)}{r_a}}$$

2432. If ABC is a triangle with usual notations prove that:

$$\frac{w_a+w_b}{w_c a^4} + \frac{w_b+w_c}{w_a b^4} + \frac{w_c+w_a}{w_b c^4} \ge \frac{2}{3R^4}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Marian Ursărescu-Romania

We prove in general: let $x, y, z > 0$, then

$$\frac{x+y}{za^4} + \frac{y+z}{xb^4} + \frac{x+z}{yc^4} \ge \frac{2}{3R^4}$$

Because $x+y \ge 2\sqrt{xy}$, then we must show that:

$$\frac{\sqrt{xy}}{za^4} + \frac{\sqrt{yz}}{xb^4} + \frac{\sqrt{xz}}{yc^4} \ge \frac{1}{3R^4}; (1)$$

$$\text{But: } \frac{\sqrt{xy}}{za^4} + \frac{\sqrt{yz}}{xb^4} + \frac{\sqrt{xz}}{yc^4} \ge 3\cdot\sqrt[3]{\frac{1}{(abc)^4}}; (2)$$

From $(1), (2)$ we must show:

$$3\cdot\sqrt[3]{\frac{1}{(abc)^4}} \ge \frac{1}{3R^4} \Leftrightarrow \frac{3^3}{(abc)^4} \ge \frac{1}{3^3R^{12}} \Leftrightarrow \frac{27}{(4Rrs)^4} \ge \frac{1}{27R^{12}} \Leftrightarrow 3\sqrt{3}R^2 \ge 4sr$$

Which is true because $R \ge 2r$ $(Euler)$ and$3\sqrt{3}R \ge 2s$ $(Mitrinovic)$.

Solution 2 by Alex Szoros-Romania

$$\sum_{cyc}\frac{w_a+w_b}{w_ca^4}=\sum_{cyc}\frac{w_a+w_b+w_c-w_a}{w_ca^4}=\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{1}{a^4w_c}\right)-\sum_{cyc}\frac{1}{a^4}=$$

$$=\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{\left(\frac{1}{a^2}\right)^2}{w_c}\right)-\sum_{cyc}\frac{1}{a^4}\geq\left(\sum_{cyc}w_a\right)\cdot\frac{\left(\sum\frac{1}{a^2}\right)^2}{\sum w_a}-\sum_{cyc}\frac{1}{a^4}=$$

$$=2\sum_{cyc}\frac{1}{b^2c^2}=\frac{2}{a^2b^2c^2}\left(\sum_{cyc}a^2\right)\geq\frac{2\cdot 9R^2}{(abc)^2};(1)$$

$$\frac{2\cdot 9R^2}{(abc)^2}\geq\frac{2}{3R^4}\Leftrightarrow 27R^6\geq(abc)^2\Leftrightarrow 3\sqrt{3}R^3\geq abc$$

$$\Leftrightarrow 3\sqrt{3}R^3\geq 4Rrs\Leftrightarrow 3\sqrt{3}R^2\geq 4sr;(2)$$

But $R\geq 2r$ $(Euler)$, so we must to show that:

$$3\sqrt{3}R^2\geq 3\sqrt{3}R\cdot 2r\geq 2s\cdot 2r;(3)\text{ because:}$$

$$3\sqrt{3}R\geq 2s\Leftrightarrow\frac{3\sqrt{3}R}{2}\geq s\ (Mitrinovic)$$

From $(1),(2),(3)$ it follows that:

$$\frac{w_a+w_b}{w_ca^4}+\frac{w_b+w_c}{w_ab^4}+\frac{w_c+w_a}{w_bc^4}\geq\frac{2}{3R^4}$$

2433. **In ΔABC the following relationship :**

$$\frac{1}{r}\sum\sqrt{r_a}\geq\sum\sqrt{\frac{2(n_a+h_a)}{(r_b-r)(r_c-r)}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:$$

$$(r_b-r)(r_c-r)=r^2\left(\frac{s}{s-b}-1\right)\left(\frac{s}{s-c}-1\right)=r^2.\frac{bc}{(s-b)(s-c)}=\frac{r^2}{\sin^2\frac{A}{2}}$$

$$\frac{\sqrt{r_a}}{r} \ge \sqrt{\frac{2(n_a+h_a)}{(r_b-r)(r_c-r)}} \leftrightarrow \frac{r_a}{\sin^2\frac{A}{2}} \ge 2(n_a+h_a) \leftrightarrow \frac{s\tan\frac{A}{2}}{2\sin^2\frac{A}{2}} - h_a \ge n_a$$

$$\leftrightarrow \left(\frac{s}{\sin A} - \frac{2sr}{a}\right)^2 \ge {n_a}^2 = s^2 - 2h_a r_a$$

$$\leftrightarrow \frac{4(R-r)^2 s^2}{a^2} \ge s^2\left(1 - 2\left(\frac{2r}{a}\right)\left(\tan\frac{A}{2}\right)\right) \leftrightarrow 4(R-r)^2$$

$$\ge (2R\sin A)^2 - 4R\sin\frac{A}{2}\cos\frac{A}{2}.4r\tan\frac{A}{2}$$

$$\leftrightarrow (R-r)^2 \ge R^2(1-\cos^2 A) - 2Rr(1-\cos A) \leftrightarrow r^2 \ge -R^2\cos^2 A + 2Rr\cos A$$

$$\leftrightarrow (R\cos A - r)^2 \ge 0, \textit{which is true}.$$

$$\to \frac{\sqrt{r_a}}{r} \ge \sqrt{\frac{2(n_a+h_a)}{(r_b-r)(r_c-r)}} \quad (\textit{And analogs})$$

$$\textit{Therefore,} \quad \frac{1}{r}\sum\sqrt{r_a} \ge \sum\sqrt{\frac{2(n_a+h_a)}{(r_b-r)(r_c-r)}}$$

2434. In ΔABC the following relationship holds:

$$\sum\frac{m_a}{a} \ge \sqrt{\left(\frac{2r}{R} - \frac{r^2}{R^2}\right)\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\textit{We have}: \ (b+c)^2 = [a+(-a+b+c)]^2 \overset{AM-GM}{\ge} 4a(-a+b+c) = 8abc.\frac{s-a}{bc}$$

$$= 32Rr.\frac{s(s-a)}{bc} = 32Rr\cos^2\frac{A}{2} \to b+c \ge 4\sqrt{2Rr}\cos\frac{A}{2} \ (\textit{and analogs}) \ \ (1)$$

$$\to \sum\frac{m_a}{a} \overset{Loscu}{\ge} \sum\frac{b+c}{2a}\cos\frac{A}{2} \overset{(1)}{\ge} \sum\frac{4\sqrt{2Rr}\cos^2\frac{A}{2}}{2.4R\sin\frac{A}{2}\cos\frac{A}{2}} = \sqrt{\frac{r}{2R}}\sum\frac{1}{\tan\frac{A}{2}}$$

$$= \sqrt{\frac{r}{2R}}\sum\frac{s}{r_a} = \sqrt{\frac{r}{2R}}.\frac{s}{r} = \frac{s}{\sqrt{2Rr}} \to \sum\frac{m_a}{a} \ge \frac{s}{\sqrt{2Rr}}$$

$$\rightarrow It\ is\ suffices\ to\ prove:\ \frac{s}{\sqrt{2Rr}} \ge \sqrt{\left(\frac{2r}{R}-\frac{r^2}{R^2}\right)\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)}\ \ (*)$$

$$Now, from\ Oppenheim's\ inequality, we\ have: \sum a^2x$$

$$\ge 4F(a,b,c).\sqrt{\sum xy}, \forall x,y,z>0$$

$$Where\ F(a,b,c)\ is\ the\ area\ of\ triangle\ with\ sides\ a,b,c.$$

$$Appliying\ this\ inequality\ for\ a\ triangle\ with\ sides\ \sqrt{a},\sqrt{b},\sqrt{c}$$

$$\rightarrow \sum \sqrt{a}^2x \overset{(2)}{\ge} 4F(\sqrt{a},\sqrt{b},\sqrt{c}).\sqrt{\sum xy}$$

$$We\ have: 16F(\sqrt{a},\sqrt{b},\sqrt{c})^2 = 2\sum(\sqrt{a}\sqrt{b})^2-\sum\sqrt{a}^4 = 2\sum ab-\sum a^2$$

$$=4r(4R+r)$$

$$\rightarrow F(\sqrt{a},\sqrt{b},\sqrt{c}) = \frac{1}{2}\sqrt{r(4R+r)} \rightarrow (2) \leftrightarrow \sum ax \ge 2\sqrt{r(4R+r)\sum xy}, \forall x,y,z>0$$

$$Let\ x=\frac{b}{a}, y=\frac{c}{b}, z=\frac{a}{c} \rightarrow \sum a.\frac{b}{a} \ge 2\sqrt{r(4R+r)\sum\frac{b}{a}.\frac{c}{b}} \leftrightarrow \sum b$$

$$\ge 2\sqrt{r(4R+r)\sum\frac{c}{a}}$$

$$\leftrightarrow s \ge \sqrt{r(4R+r)\sum\frac{a}{b}}, similarly:\ s \ge \sqrt{r(4R+r)\sum\frac{b}{a}}$$

$$\rightarrow \sqrt{\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)} \le \frac{s^2}{r(4R+r)} \overset{Blundon-Gerretsen}{\le} \frac{s}{r}.\sqrt{\frac{R}{2(2R-r)}} = \frac{s}{\sqrt{2Rr}}.\sqrt{\frac{R^2}{2Rr-r^2}}$$

$$\rightarrow \frac{s}{\sqrt{2Rr}} \ge \sqrt{\frac{2Rr-r^2}{R^2}\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)} \leftrightarrow \frac{s}{\sqrt{2Rr}} \ge \sqrt{\left(\frac{2r}{R}-\frac{r^2}{R^2}\right)\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)}$$

$$\rightarrow (*)\ is\ true.$$

$$Therefore,$$

$$\sum\frac{m_a}{a} \ge \sqrt{\left(\frac{2r}{R}-\frac{r^2}{R^2}\right)\left(\sum\frac{a}{b}\right)\left(\sum\frac{b}{a}\right)}$$

2435 . $In\ \Delta ABC, V - Bevan's\ point, I - incenter. Prove\ that:$

$$[AIV]+[BIV]+[CIV]<3R(R-r)$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

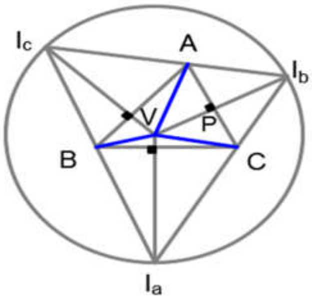
$$[AIV]=\frac{1}{2}AI.AV.\sin IAV<\frac{1}{2}AI.AV\overset{AM-GM}{\leq}\frac{1}{4}(AI^2+AV^2)\ (and\ analogs)$$

$$\rightarrow [AIV]+[BIV]+[CIV]<\frac{1}{4}\left(\sum AI^2+\sum AV^2\right)$$

$$We\ have: \sum AI^2=r^2\sum\frac{1}{\sin^2\frac{A}{2}}=r^2\sum\frac{bc}{(s-b)(s-c)}=\frac{1}{s}\sum bc(s-a)=$$

$$=(s^2+r^2+4Rr)-3.4Rr=s^2+r^2-8Rr\rightarrow\sum AI^2=s^2+r^2-8Rr\ \ (2)$$

$$Now, let\ us\ prove: \sum AV^2=12R^2-s^2-r^2-4Rr$$



$$Let\ P\ be\ the\ feet\ of\ the\ perpendicular\ from\ I_b\ to\ AC.$$

$$We\ know\ that: P\in VI_b, AP=s-c, PI_b=r_b\ and\ VI_a=VI_b=VI_c=2R$$

$$\rightarrow AV^2=AP^2+PV^2=(s-c)^2+(2R-r_b)^2\ \ (and\ analogs)$$

$$\rightarrow\sum AV^2=\sum(s-c)^2+\sum(2R-r_b)^2$$

$$=3s^2-2s\sum c+\sum c^2+12R^2-4R\sum r_b+\sum {r_b}^2$$

$$=3s^2-4s^2+2(s^2-r^2-4Rr)+12R^2-4R(4R+r)+(4R+r)^2-2s^2$$

$$\rightarrow\sum AV^2=12R^2-s^2-r^2-4Rr\ \ (2)$$

$$\rightarrow [AIV]+[BIV]+[CIV]<\frac{1}{4}\left(\sum AI^2+\sum AV^2\right)=\frac{1}{4}(12R^2-12Rr)=3R(R-r).$$

2436. In ΔABC the following relationship holds :

$$\sqrt{\left(\prod\cos\frac{A}{2}\right)\left(\sum\frac{w_b+w_c}{a}\right)}\ge\frac{\sqrt{2}}{2}\cdot\frac{s}{R}$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(*)\quad\sqrt{\left(\prod\cos\frac{A}{2}\right)\left(\sum\frac{w_b+w_c}{a}\right)}\ge\frac{\sqrt{2}}{2}\cdot\frac{s}{R}$$

$$We\ have:\ \sum\frac{w_b+w_c}{a}=\sum w_a\left(\frac{1}{b}+\frac{1}{c}\right)=\sum\frac{2bc}{b+c}.\cos\frac{A}{2}.\frac{b+c}{bc}=2\sum\cos\frac{A}{2}$$

$$\to(*)\leftrightarrow\sqrt{\left(\prod\cos\frac{A}{2}\right)\left(\sum\cos\frac{A}{2}\right)}\ge\frac{1}{2}.\frac{s}{R}=2\prod\cos\frac{A}{2}\leftrightarrow\sum\cos\frac{A}{2}\ge4\prod\cos\frac{A}{2}$$

$$\leftrightarrow\sum\cos\frac{\pi-2A}{2}\ge4\prod\cos\frac{\pi-2A}{2}\ for\ any\ triangle\ with\ angles\ \pi-2A,\pi-2B,\pi-2C.$$

$$\leftrightarrow\sum\sin A\ge4\prod\sin A\leftrightarrow\frac{s}{R}\ge4.\frac{sr}{2R^2}\ \leftrightarrow\ R\ge2r\ (Euler)$$

$$Therefore,\quad\sqrt{\left(\prod\cos\frac{A}{2}\right)\left(\sum\frac{w_b+w_c}{a}\right)}\ge\frac{\sqrt{2}}{2}\cdot\frac{s}{R}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\prod_{cyc}\cos\frac{A}{2}=\frac{s}{4R};\ \sum_{cyc}\frac{w_b+w_c}{a}=\sum_{cyc}w_a\left(\frac{1}{b}+\frac{1}{c}\right)=\sum_{cyc}w_a\cdot\frac{b+c}{bc}=$$

$$=\sum_{cyc}\frac{2bc}{b+c}\cdot\cos\frac{A}{2}\cdot\frac{b+c}{bc}=2\cdot\sum_{cyc}\cos\frac{A}{2}\overset{AM-GM}{\ge}6\cdot\sqrt[3]{\prod_{cyc}\cos\frac{A}{2}}=6\cdot\sqrt[3]{\frac{s}{4R}}$$

So, we need to prove that:

$$\sqrt{\frac{s}{4R}}\cdot\sqrt{6}\cdot\sqrt[6]{\frac{s}{4R}}\ge\frac{\sqrt{2}}{2}\cdot\frac{s}{R}\Leftrightarrow\sqrt{6}\cdot\sqrt{\frac{s}{R}}\cdot\sqrt[6]{\frac{s}{6R}}\ge\sqrt{2}\cdot\frac{s}{R}\Leftrightarrow$$

$$\left(\frac{s}{R}\right)^3\cdot6^3\cdot\frac{s}{4R}\ge8\cdot\left(\frac{s}{R}\right)^6\Leftrightarrow\frac{6^3}{4\cdot8}\ge\left(\frac{s}{R}\right)^2\Leftrightarrow\frac{6\sqrt{6}}{2}\ge2\sqrt{2}\cdot\frac{s}{R}$$

$$\Leftrightarrow\frac{3\sqrt{3}}{2}\ge\frac{s}{R}\ (Mitrinovic)$$

2437. In any ΔABC the following relationship holds:

$$\left(\sum ab\right)^2+2\left(\sum a^2\right)^2\ge 16\sqrt{3}s^3r$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\sum ab\right)^2+2\left(\sum a^2\right)^2\overset{\sum a^2\ge\sum ab}{\ge}2\left(\sum ab\right)^2+\left(\sum a^2\right)^2\overset{CBS}{\ge}\frac{1}{3}\left(2\sum ab+\sum a^2\right)^2=$$

$$=\frac{1}{3}\left(\sum a\right)^4=\frac{16}{3}s^4\overset{Mitrinovic}{\ge}\frac{16}{3}.3\sqrt{3}rs^3$$

$$Therefore,\quad \left(\sum ab\right)^2+2\left(\sum a^2\right)^2\ge 16\sqrt{3}s^3r$$

2438. In any Δ ABC the following relationship holds:

$$3\sum\frac{1}{a^3}\tan\frac{A}{2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{1}{a^3}\cot\frac{A}{2}=s\sum\left(\frac{1}{a^3}\left(\frac{s-a}{rs}\right)\right)$$

$$=\frac{1}{r}\left(\frac{s}{64R^3r^3s^3}\left(\left(\sum ab\right)^3-3.4Rrs.2s(s^2+2Rr+r^2)\right)\right.$$

$$\left.-\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right)$$

$$=\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)-4Rr((s^2+4Rr+r^2)^2-16Rrs^2)}{64R^3r^4s^2}$$

$$=\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}\overset{(i)}{=}\sum\frac{1}{a^3}\cot\frac{A}{2}$$

$$\sum\frac{1}{a^3}\tan\frac{A}{2}=\sum\left(\frac{1}{a^2}\left(\frac{\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\right)\right)=\frac{1}{4R}\sum\left(\frac{1}{a^2}\left(1+\tan^2\frac{A}{2}\right)\right)$$

$$=\frac{1}{4R}\left(\sum\frac{1}{a^2}+\sum\frac{\tan^2\frac{A}{2}}{16R^2\cos^4\frac{A}{2}\tan^2\frac{A}{2}}\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\sum\left(1+\tan^2\frac{A}{2}\right)^2\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\sum\left(1+\tan^4\frac{A}{2}+2\tan^2\frac{A}{2}\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(\left(\sum r_a^2\right)^2-2\sum r_a^2r_b^2\right)+\frac{2}{s^2}\left(\sum r_a^2\right)\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}\right.$$

$$+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(((4R+r)^2-2s^2)^2-2\left(s^4-2rs^2(4R+r)\right)\right)\right.$$

$$\left.\left.+\frac{2}{s^2}((4R+r)^2-2s^2)\right)\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{64R^3r^2s^2}$$

$$+\frac{3s^4+2s^2(4R+r)^2-4s^4+(4R+r)^4-4s^2(4R+r)^2+4s^4-2s^4+4rs^2(4R+r)}{64R^3s^4}$$

$$=\frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{64R^3r^2s^4}\overset{(ii)}{=}\sum\frac{1}{a^3}\tan\frac{A}{2}\ \therefore\ (i),(ii)$$

$$\Rightarrow 3\sum\frac{1}{a^3}\tan\frac{A}{2}\le\sum\frac{1}{a^3}\cot\frac{A}{2}$$

$$\Leftrightarrow\frac{3s^6-3(8Rr-3r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4}{64R^3r^2s^4}$$

$$\le\frac{s^6-(16Rr-3r^2)s^4+r^2s^2(32R^2-8Rr+3r^2)+r^4(4R+r)^2}{64R^3r^4s^2}$$

$$\Leftrightarrow s^8-16Rrs^6+(32R^2+16Rr-6r^2)r^2s^4+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4\overset{(1)}{\geq}0$$

$$\text{Now, LHS of (1)}\overset{\text{Gerretsen}}{\geq}-5r^2s^6+(32R^2+16Rr-6r^2)r^2s^4$$

$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$

$$\overset{\text{Gerretsen}}{\geq} r^2s^4\left(32R^2+16Rr-6r^2-5(4R^2+4Rr+3r^2)\right)$$

$$+r^4(64R^2-16Rr-8r^2)s^2-3r^4(4R+r)^4$$

$$\overset{\text{Gerretsen}}{\geq} r^2s^2\left((12R^2-4Rr-21r^2)(16Rr-5r^2)+r^2(64R^2-16Rr-8r^2)\right)$$

$$-3r^4(4R+r)^4$$

$$=r^3s^2(192R^3-60R^2r-332Rr^2+97r^3)$$

$$-3r^4(4R+r)^4 \overset{\text{Gerretsen}}{\geq} r^4\left((192R^3-60R^2r-332Rr^2+97r^3)(16R-5r)-3(4R+r)^4\right)$$

$$\overset{?}{\geq} 0 \Leftrightarrow 576t^4-672t^3-1325t^2+791t-122 \overset{?}{\geq} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(576t^3+480t^2-365t+61) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (1) \text{ is true} \therefore 3\sum\frac{1}{a^3}\tan\frac{A}{2} \leq \sum\frac{1}{a^3}\cot\frac{A}{2} \text{ (QED)}$$

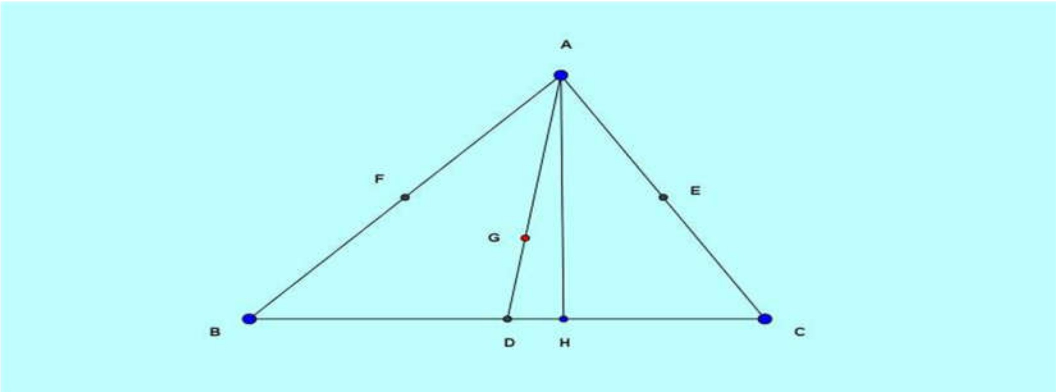
2439. **In any Δ ABC the following relationship holds:**

$$\sum\frac{r_a+r}{r_a-r} \geq \sum\frac{4F+(b-c)^2}{a.s_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

Proof :



$$AD = m_a \text{ and } AH = h_a \therefore BH = c\cos B \Rightarrow DH = c\cos B - \frac{a}{2} \text{ and } CH = b\cos C \Rightarrow DH = \frac{a}{2} - b\cos C \therefore 2DH = c\cos B - b\cos C$$

Here $c > b$ and proceeding in a similar manner when $b > c$, $2DH = b\cos C - c\cos B \therefore 2DH = |b\cos C - c\cos B|$

$$= \left|b\left(\frac{a^2+b^2-c^2}{2ab}\right) - c\left(\frac{c^2+a^2-b^2}{2ca}\right)\right| = \frac{2|b^2-c^2|}{2a} \Rightarrow 4DH^2 = \frac{(b^2-c^2)^2}{a^2}$$

$$\Rightarrow AD^2 - AH^2 = \frac{(b^2-c^2)^2}{4a^2} \Rightarrow m_a^2 - h_a^2 = \frac{(b^2-c^2)^2}{4a^2}$$

$$\Rightarrow \frac{m_a^2-h_a^2}{h_a^2} = \frac{(b^2-c^2)^2}{4a^2\left(\frac{4F^2}{a^2}\right)} \Rightarrow \frac{m_a^2-h_a^2}{h_a^2} \overset{(a)}{=} \frac{(b^2-c^2)^2}{16F^2}$$

$$\frac{am_a}{F} \ge 2 + \frac{(b-c)^2}{2F} \Leftrightarrow \frac{2m_a}{h_a} - 2 \ge \frac{(b-c)^2}{2F} \Leftrightarrow \left(\frac{m_a}{h_a} - 1\right)\left(\frac{m_a}{h_a} + 1\right) \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a} + 1\right) \Leftrightarrow \frac{m_a^2-h_a^2}{h_a^2} \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a} + 1\right)$$

$$\overset{\text{via }(a)}{\Leftrightarrow} \frac{(b^2-c^2)^2}{16F^2} \ge \frac{(b-c)^2}{4F}\left(\frac{m_a}{h_a} + 1\right) \Leftrightarrow \frac{(b+c)^2}{4F} \ge \frac{m_a}{h_a} + 1 = \frac{am_a}{2F} + 1 = \frac{2am_a + 4F}{4F}$$

$$\Leftrightarrow (b+c)^2 - 2am_a \ge 4F$$

$$\Leftrightarrow ((b+c)^2 - 2am_a)^2 \ge 16F^2 \left(\because 2am_a < 2(b+c)\left(\frac{b+c}{2}\right) \Rightarrow (b+c)^2 > 2am_a \Rightarrow (b+c)^2 - 2am_a > 0\right)$$

$$\Leftrightarrow (b+c)^4 + 4a^2m_a^2 - 4am_a(b+c)^2 \ge 2\sum a^2b^2 - \sum a^4$$

$$\Leftrightarrow (b+c)^4 + a^2(2b^2+2c^2-a^2) + \sum a^4 - 2\sum a^2b^2 \ge 4am_a(b+c)^2$$

$$\Leftrightarrow b^4 + c^4 + 2b^2c^2 + 2b^3c + 2bc^3 \ge 2am_a(b+c)^2$$

$$\Leftrightarrow (b^2+c^2)^2 + 2bc(b^2+c^2) \ge 2am_a(b+c)^2 \Leftrightarrow (b^2+c^2)(b+c)^2 \ge 2am_a(b+c)^2$$

$$\Leftrightarrow b^2 + c^2 \ge 2am_a \Leftrightarrow (b^2+c^2)^2 \ge 4a^2m_a^2$$

$$\Leftrightarrow b^4+c^4+2b^2c^2\geq a^2(2b^2+2c^2-a^2)\Leftrightarrow b^4+c^4+a^4+2b^2c^2-2a^2b^2-2c^2a^2$$
$$\geq 0\Leftrightarrow (b^2+c^2-a^2)^2\geq 0\Leftrightarrow \cos^2 A\geq 0$$
$$\to \text{true}\Rightarrow \frac{am_a}{F}\geq 2+\frac{(b-c)^2}{2F}\Rightarrow 2am_a\geq 4F+(b-c)^2\Rightarrow \frac{4F+(b-c)^2}{am_a}\leq 2$$
$$\Rightarrow \frac{4F+(b-c)^2}{a\left(\frac{2bc}{b^2+c^2}m_a\right)}\leq \frac{b^2+c^2}{bc}$$
$$\Rightarrow \frac{4F+(b-c)^2}{as_a}\leq \frac{b}{c}+\frac{c}{b}\text{ and analogs}\Rightarrow \sum\frac{4F+(b-c)^2}{as_a}\leq \sum\frac{b}{c}+\sum\frac{c}{b}$$
$$=\sum\frac{c}{a}+\sum\frac{b}{a}=\sum\frac{b+c}{a}=\sum\frac{\frac{rs}{s-a}+\frac{rs}{s}}{\frac{rs}{s-a}-\frac{rs}{s}}$$
$$=\sum\frac{r_a+r}{r_a-r}\therefore \sum\frac{r_a+r}{r_a-r}\geq \sum\frac{4F+(b-c)^2}{a.s_a}\ (\text{QED})$$

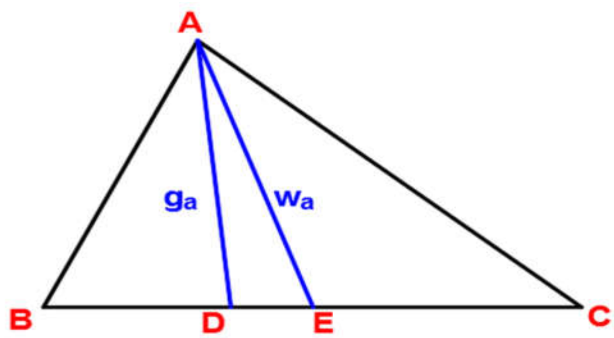
2440. In ΔABC, g_a – Gergonne's cevian, prove that :

$$a.\sum w_a-\sum g_a\leq \sum\frac{(b+c-a)[b-c]}{b+c}.$$
$$b.\sum {h_a}^2+\frac{2r(R^2-4r^2)}{R}\leq s^2.$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

a.



$$We\ know\ that: BD = s - b\ and\ BE = \frac{ac}{b+c} \to DE = \left|\frac{ac}{b+c} - (s-b)\right|$$

$$= \frac{|2ac - (b+c)(a-b+c)|}{2(b+c)} =$$

$$= \frac{|a(c-b) + b^2 - c^2|}{2(b+c)} = \frac{(b+c-a)|b-c|}{2(b+c)} \to DE = \frac{(b+c-a)|b-c|}{2(b+c)}$$

$$We\ have: w_a - g_a \overset{\Delta}{\leq} DE = \frac{(b+c-a)|b-c|}{2(b+c)}\ (and\ analogs)$$

$$Therefore, \sum w_a - \sum g_a \leq \sum \frac{(b+c-a)|b-c|}{2(b+c)} \leq \sum \frac{(b+c-a)|b-c|}{(b+c)}$$

$b.$ $we\ know\ that: h_a \leq g_a\ and\ {g_a}^2 = (s-a)^2 + 2rh_a\ (and\ analogs).$

$$\to \sum {h_a}^2 \leq \sum {g_a}^2 = \sum (s-a)^2 + 2r \sum h_a$$

$$= 3s^2 - 4s^2 + 2(s^2 - r^2 - 4Rr) + \frac{r(s^2 + r^2 + 4Rr)}{R} =$$

$$= s^2 + r.\frac{s^2 + r^2 + 4Rr - 2R(r+4R)}{R} \overset{Gerretsen}{\leq} s^2$$

$$+ r.\frac{(4R^2 + 4Rr + 3r^2) + r^2 + 2Rr - 8R^2}{R} =$$

$$= s^2 - \frac{2r(2R^2 - 3Rr - 2r^2)}{R} \overset{?}{\leq} s^2 - \frac{2r(R^2 - 4r^2)}{R} \leftrightarrow 2R^2 - 3Rr - 2r^2 \geq R^2 - 4r^2$$

$\leftrightarrow R^2 - 3Rr + 2r^2 \geq 0 \leftrightarrow (R-2r)(R-r) \geq 0,$ $which\ is\ true\ from\ Euler\ (R \geq 2r).$

$$Therefore, \quad \sum {h_a}^2 + \frac{2r(R^2 - 4r^2)}{R} \leq s^2$$

2441. ***In ΔABC the following relationship holds:***

$$\sum \frac{m_a}{h_a} \geq \sum \frac{a}{b+c} + \sum \frac{m_a}{m_b + m_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$From\ CBS, we\ have: \frac{1}{b+c} \leq \frac{1}{4}\left(\frac{1}{b} + \frac{1}{c}\right) \to \frac{a}{b+c} \leq \frac{1}{4}\left(\frac{a}{b} + \frac{a}{c}\right)\ (and\ analogs)$$

$$Similarly,\quad \frac{m_a}{m_b+m_c}\le\frac{1}{4}\left(\frac{m_a}{m_c}+\frac{m_a}{m_b}\right)(and\ analogs)$$

$$\to\sum\frac{a}{b+c}+\sum\frac{m_a}{m_b+m_c}\le\frac{1}{4}\left(\sum\left(\frac{a}{b}+\frac{a}{c}\right)+\sum\left(\frac{m_a}{m_c}+\frac{m_a}{m_b}\right)\right)\overset{?}{\le}\sum\frac{m_a}{h_a}\ (*)$$

$$We\ know\ that:$$

$$m_a\overset{Tereshin}{\ge}\frac{b^2+c^2}{4R}=\frac{1}{2}\cdot\frac{bc}{2R}\cdot\frac{b^2+c^2}{bc}=\frac{h_a}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\leftrightarrow\frac{b}{c}+\frac{c}{b}\le\frac{2m_a}{h_a}\quad(1)$$

$$m_a,m_b,m_c\ can\ be\ length\ sides\ of\ triangle\ with:$$

$$F_m=\frac{3}{4}F,\overline{m_a}=\frac{3}{4}a,\overline{h_a}=\frac{2F_m}{m_a}=\frac{3F}{2m_a}$$

$$In\ \Delta m_am_bm_c,(1)\leftrightarrow$$

$$\frac{m_b}{m_c}+\frac{m_c}{m_b}\le\frac{2\overline{m_a}}{\overline{h_a}}\leftrightarrow\frac{m_b}{m_c}+\frac{m_c}{m_b}\le\frac{3}{2}a.\frac{2m_a}{3F}=\frac{2m_a}{h_a}\to\frac{m_b}{m_c}+\frac{m_c}{m_b}\le\frac{2m_a}{h_a}(2)$$

$$(1),(2)\to\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\le\frac{4m_a}{h_a}\quad(and\ analogs)$$

$$\to\sum\left(\frac{b}{c}+\frac{c}{b}\right)+\sum\left(\frac{m_b}{m_c}+\frac{m_c}{m_b}\right)\le4\sum\frac{m_a}{h_a}\leftrightarrow$$

$$\frac{1}{4}\left(\sum\left(\frac{a}{b}+\frac{a}{c}\right)+\sum\left(\frac{m_a}{m_c}+\frac{m_a}{m_b}\right)\right)\le\sum\frac{m_a}{h_a}$$

$$\to(*)\ is\ true. Therefore,\sum\frac{m_a}{h_a}\ge\sum\frac{a}{b+c}+\sum\frac{m_a}{m_b+m_c}$$

2442. In ΔABC the following relationship holds:

$$h_a+R\ge w_a+2r$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

First, we prove that:

$$\left(\frac{w_a}{h_a}\right)^2\le\frac{R}{2r};(1)$$

We have: $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ **and** $h_a=w_a\cos\frac{B-C}{2};(2)$

From (2)**, relations** (1) **becomes:** $\frac{1}{\cos^2\left(\frac{B-C}{2}\right)}\le\frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\Leftrightarrow$

$$\cos^2\left(\frac{B-C}{2}\right)\ge 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Leftrightarrow \cos^2\left(\frac{B-C}{2}\right)\ge 4\sin\frac{A}{2}\left(\cos\frac{B-C}{2}-\sin\frac{A}{2}\right)$$

$$\left(\cos\frac{B-C}{2}-2\sin\frac{A}{2}\right)^2\ge 0\to(1)\text{ is true.}$$

We must show that: $w_a-h_a\le R-2r\Leftrightarrow$

$h_a\left(\frac{w_a}{h_a}-1\right)\le 2r\left(\frac{R}{2r}-1\right)$. But from (1): $\frac{w_a}{h_a}\le\sqrt{\frac{R}{2r}}$, we must show that:

$$h_a\left(\sqrt{\frac{R}{2r}}-1\right)\le 2r\left(\sqrt{\frac{R}{2r}}-1\right)\left(\sqrt{\frac{R}{2r}}+1\right);(2)$$

Case I. If ΔABC is equilateral, then $R=2r$, true.

Case II. If ΔABC is non-equilateral, then we must prove:

$h_a\le 2r\left(\sqrt{\frac{R}{2r}}+1\right)$. From $R\ge 2r$ $(Euler)$ we must to prove that:

$$h_a\le 2r\left(\sqrt{\frac{R}{2r}}+1\right);(3)\Leftrightarrow h_a\le\sqrt{2Rr}+2r\Leftrightarrow h_a-2r\le\sqrt{2Rr}\Leftrightarrow$$

$$\frac{2F}{a}-\frac{2F}{s}\le\sqrt{2\cdot\frac{abc}{4F}\cdot\frac{F}{s}}\Leftrightarrow\frac{2F(s-a)}{as}\le\sqrt{\frac{abc}{2s}}\Leftrightarrow$$

$$2\sqrt{2}\cdot\sqrt{s}\cdot(s-a)\cdot\sqrt{s(s-a)(s-b)(s-c)}\le as\sqrt{abc}\Leftrightarrow$$

$$2\sqrt{2}(s-a)\cdot\sqrt{(s-a)(s-b)(s-c)}<a\sqrt{abc};(4)$$

Let: $s-a=x, s-b=y, s-c=z$, then $a=y+z, b=z+x, c=x+y$ and $s=x+y+z$.

Thus, $(4)\Leftrightarrow 2\sqrt{2}\cdot x\cdot\sqrt{xyz}\le(y+z)\sqrt{(x+y)(y+z)(z+x)};(5)$

But: $x+y\ge 2\sqrt{xy}, y+z\ge 2\sqrt{yz}, z+x\ge 2\sqrt{zx}$, hence

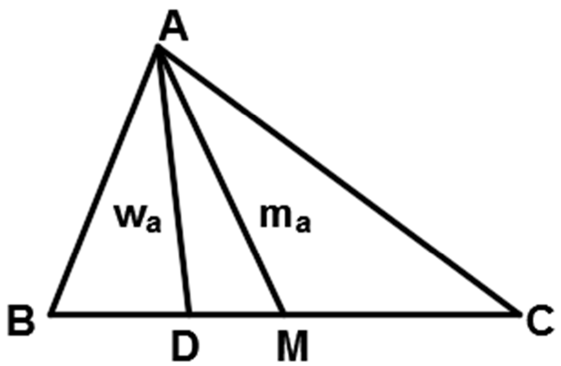
$\sqrt{(x+y)(y+z)(z+x)}\ge 2\sqrt{2}\cdot\sqrt{xyz}$ and $y+z>x$, then (5) is true, then (3) is true.

2443. In ΔABC the following relationship holds:

$$\sum\frac{|h_b-h_c|}{b+c}\ge\frac{1}{R}\sum(m_a-w_a)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco



$$We\ know\ that: DB=\frac{ac}{b+c}\to DM=|BM-DB|=\left|\frac{a}{2}-\frac{ac}{b+c}\right|=\frac{a|b-c|}{2(b+c)}$$

$$w_a+DM\overset{\Delta}{\geq}m_a\leftrightarrow w_a+\frac{a|b-c|}{2(b+c)}\geq m_a\ \leftrightarrow \frac{abc\left|\frac{1}{c}-\frac{1}{b}\right|}{2(b+c)}\geq m_a-w_a\leftrightarrow$$

$$\frac{\left|\frac{2F}{c}-\frac{2F}{b}\right|}{b+c}\geq\frac{m_a-w_a}{R}\leftrightarrow\frac{|h_b-h_c|}{b+c}\geq\frac{m_a-w_a}{R}\ (and\ analogs)$$

$$Therefore,\qquad \sum\frac{|h_b-h_c|}{b+c}\geq\frac{1}{R}\sum(m_a-w_a)$$

2444. In any Δ ABC the following relationship holds:

$$\sum\frac{1}{a^3}\tan\frac{A}{2}\leq\frac{1}{12Rr^2}\leq\frac{1}{24r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum\frac{1}{a^3}\tan\frac{A}{2}=\sum\left(\frac{1}{a^2}\left(\frac{\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}\right)\right)=\frac{1}{4R}\sum\left(\frac{1}{a^2}\left(1+\tan^2\frac{A}{2}\right)\right)$$

$$=\frac{1}{4R}\left(\sum\frac{1}{a^2}+\sum\frac{\tan^2\frac{A}{2}}{16R^2\cos^4\frac{A}{2}\tan^2\frac{A}{2}}\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\sum\left(1+\tan^2\frac{A}{2}\right)^2\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\sum\left(1+tan^4\frac{A}{2}+2tan^2\frac{A}{2}\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(\left(\sum r_a^2\right)^2-2\sum r_a^2r_b^2\right)+\frac{2}{s^2}\left(\sum r_a^2\right)\right)\right)$$

$$=\frac{1}{4R}\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{16R^2r^2s^2}+\frac{1}{16R^2}\left(3+\frac{1}{s^4}\left(((4R+r)^2-2s^2)^2-2\left(s^4-2rs^2(4R+r)\right)\right)+\frac{2}{s^2}((4R+r)^2-2s^2)\right)\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{64R^3r^2s^2}$$

$$+\frac{3s^4+2s^2(4R+r)^2-4s^4+(4R+r)^4-4s^2(4R+r)^2+4s^4-2s^4+4rs^2(4R+r)}{64R^3s^4}$$

$$=\frac{s^6-(8Rr-3r^2)s^4-r^2s^2(16R^2-8Rr-3r^2)+r^2(4R+r)^4}{64R^3r^2s^4}\le\frac{1}{12Rr^2}$$

$$\Leftrightarrow 3s^6-(16R^2+24Rr-9r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4\overset{(i)}{\le}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where m

$$=2R^2+10Rr-r^2 \text{ and } n=2(R-2r)\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\le 0$$

$$\Leftrightarrow 3s^6-3s^4(4R^2+20Rr-2r^2)+3rs^2(4R+r)^3\le 0$$

$\Rightarrow$ in order to prove (i), it suffices to prove :

$$3s^6-(16R^2+24Rr-9r^2)s^4-3r^2s^2(16R^2-8Rr-3r^2)+3r^2(4R+r)^4$$
$$\le 3s^6-3s^4(4R^2+20Rr-2r^2)+3rs^2(4R+r)^3$$

$$\Leftrightarrow (4R^2-36Rr-3r^2)s^4+3rs^2(64R^3+64R^2r+4Rr^2-2r^3)-3r^2(4R+r)^4\ge 0$$

$$\Leftrightarrow 4(R-2r)^2s^4-(20Rr+19r^2)s^4+3rs^2(64R^3+64R^2r+4Rr^2-2r^3)-3r^2(4R+r)^4\overset{(ii)}{\ge}0$$

Now, LHS of (ii) $\overset{\text{Gerretsen}}{\ge}$ $\Big(4(16Rr-5r^2)(R-2r)^2$

$$-(20Rr+19r^2)(4R^2+4Rr+3r^2)$$
$$+3r(64R^3+64R^2r+4Rr^2-2r^3)\Big)s^2$$

$$-3r^2(4R+r)^4=r(176R^3-240R^2r+212Rr^2-143r^3)s^2-3r^2(4R+r)^4$$

$$=r\left((R-2r)(176R^2+112Rr+436r^2)+729r^3\right)s^2$$

$$-3r^2(4R+r)^4 \overset{Gerretsen}{\gtrless} r\left((R-2r)(176R^2+112Rr+436r^2)+729r^3\right)(16Rr-5r^2) - 3r^2(4R+r)^4$$

$$\Leftrightarrow 512t^4-1372t^3+1076t^2-849t+178 \overset{?}{\gtrless} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(512t^2+676t+1732)+3375\right) \overset{?}{\gtrless} 0 \to \text{true}$$

$$\because t \overset{Euler}{\gtrless} 2 \Rightarrow (ii) \Rightarrow (i) \text{ is true} \therefore \sum \frac{1}{a^3}\tan\frac{A}{2} \le \frac{1}{12Rr^2} \overset{Euler}{\lesssim} \frac{1}{24r^3} \text{ (QED)}$$

2445. If ABC is a nonisosceles triangle,and $x,y,z>0$, then prove that:

$$\sum \frac{a^8}{(xa+yb+zc)(a-b)^2(a-c)^2} > \frac{288\sqrt{3}r^3}{x+y+z}$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum \frac{a^8}{(xa+yb+zc)(a-b)^2(a-c)^2}$$

$$= \sum \frac{\left(\frac{a^4}{|a-b||a-c|}\right)^2}{xa+yb+zc} \overset{CBS}{\ge} \frac{\left(\sum \frac{a^4}{|a-b||a-c|}\right)^2}{\sum(xa+yb+zc)} =$$

$$= \frac{1}{2s(x+y+z)}\left(\sum\left|\frac{a^4}{(a-b)(a-c)}\right|\right)^2 \overset{\Delta}{\ge} \frac{1}{2s(x+y+z)}\left|\sum \frac{a^4}{(a-b)(a-c)}\right|^2$$

$$Now, we\ have: \frac{a^4}{(a-b)(a-c)} = a^4\left(-\frac{1}{(b-a)(b-c)} - \frac{1}{(c-a)(c-b)}\right)$$

$$\to \sum \frac{a^4}{(a-b)(a-c)} = \frac{b^4-a^4}{(b-a)(b-c)} + \frac{c^4-a^4}{(c-a)(c-b)} =$$

$$= \frac{(b-a)(b+a)(b^2+a^2)}{(b-a)(b-c)} + \frac{(c-a)(c+a)(c^2+a^2)}{(c-a)(c-b)}$$

$$= \frac{(b+a)(b^2+a^2)-(c+a)(c^2+a^2)}{b-c} =$$

$$=\frac{b^3-c^3+a^2b-a^2c+ab^2-ac^2}{b-c}=\frac{(b-c)(a^2+b^2+c^2+ab+bc+ca)}{b-c}$$

$$=\sum a^2+\sum ab=$$

$$=\frac{1}{2}\sum(a+b)^2\overset{CBS}{\geq}\frac{1}{2.3}\left[\sum(a+b)\right]^2=\frac{8s^2}{3}\overset{Mitrinovix}{\geq}\frac{8.3\sqrt{3}sr}{3}=8\sqrt{3}S$$

$$\to\frac{1}{2s(x+y+z)}\left|\sum\frac{a^4}{(a-b)(a-c)}\right|^2\geq\frac{\left(8\sqrt{3}S\right)^2}{2s\sum x}=\frac{96sr^2}{x+y+z}\overset{Mitrinovix}{\geq}\frac{96.3\sqrt{3}r^3}{x+y+z}$$

$$Therefore,\sum\frac{a^8}{(xa+yb+zc)(a-b)^2(a-c)^2}\geq\frac{288\sqrt{3}r^3}{x+y+z}$$

2446. **In any Δ ABC the following relationship holds:**

$$\sum\frac{a}{\sqrt{h_a-2r}}\geq\sqrt{R}\sum\frac{n_a}{h_a}+\frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$b+c-a=4R\cos\frac{A}{2}\cos\frac{B-C}{2}-4R\sin\frac{A}{2}\cos\frac{A}{2}=4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-\cos\frac{B+C}{2}\right)$$

$$=8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow s-a\overset{(1)}{=}4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{a}{\sqrt{h_a-2r}}=\frac{a}{\sqrt{\frac{2rs}{a}-2r}}=\frac{a}{\sqrt{2r}}\sqrt{\frac{a}{s-a}}\overset{via\ (1)}{=}\frac{a}{\sqrt{2r}}\sqrt{\frac{4R\cos\frac{A}{2}\sin^2\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}$$

$$=\frac{a\sin\frac{A}{2}}{\sqrt{2r}}\sqrt{\frac{4R}{r}}=\frac{\sqrt{2R}}{r}a\sin\frac{A}{2}\ and\ analogs\overset{summing\ up}{\Rightarrow}$$

$$\sum\frac{a}{\sqrt{h_a-2r}}=\frac{\sqrt{2R}}{r}\sum a\sin\frac{A}{2}\geq\frac{s\sqrt{2R}}{r}\Leftrightarrow\sum\cos\frac{A}{2}\sin^2\frac{A}{2}$$

$$\geq\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\ \text{applying which on a triangle with angles}\ \pi-2A,$$

$$\pi-2B,\pi-2C,\text{we get}:\sum\cos\frac{\pi-2A}{2}\sin^2\frac{\pi-2A}{2}$$

$$\geq\cos\frac{\pi-2A}{2}\cos\frac{\pi-2B}{2}\cos\frac{\pi-2B}{2}\Leftrightarrow\sum\sin A\cos^2A\geq\sin A\sin B\sin C$$

$$\Leftrightarrow\sum\sin A(1-\sin^2A)\geq\frac{4Rrs}{8R^3}\Leftrightarrow\sum\frac{a}{2R}-\sum\frac{a^3}{8R^3}\geq\frac{4Rrs}{8R^3}$$

$$\Leftrightarrow\frac{s}{R}-\frac{2s(s^2-6Rr-3r^2)}{8R^3}-\frac{4Rrs}{8R^3}\geq 0$$

$$\Leftrightarrow\frac{8R^2-2(s^2-6Rr-3r^2)-4Rr}{8R^3}\geq 0\Leftrightarrow s^2\leq 4R^2+8Rr+3r^2\rightarrow\text{true (Gerretsen)}$$

$$\therefore\sum\frac{a}{\sqrt{h_a-2r}}\overset{(a)}{\geq}\frac{s\sqrt{2R}}{r}$$

$$r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(i)}{=}4R\cos^2\frac{A}{2}$$

$$\text{Now, }(b+c)^2\geq 32Rr\cos^2\frac{A}{2}\overset{\text{by (i)}}{=}8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow(b+c)^2+4a^2-4a(b+c)\geq 0\Leftrightarrow(b+c-2a)^2\geq 0\rightarrow\text{true}\therefore b+c$$

$$\geq 4\sqrt{2Rr}\cos\frac{A}{2}\ \text{and analogs}\Rightarrow\sum m_a\overset{\text{Ioscu}}{\geq}\sum\frac{b+c}{2}\cos\frac{A}{2}$$

$$\geq \sqrt{2Rr}\sum\left(2\cos^2\frac{A}{2}\right) = \sqrt{2Rr}\sum(1+\cos A) = \sqrt{2Rr}\left(\frac{4R+r}{R}\right) = \sqrt{\frac{2r}{R}(4R+r)}$$

$$\Rightarrow \frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)}$$

$$\leq \sqrt{\frac{R}{2r}}\sum\sqrt{2\left(\frac{rs}{s-a}-\frac{rs}{s}\right)} = \sqrt{R}\sum\sqrt{\frac{a}{s-a}} = \sqrt{R}\sum\sqrt{\frac{4R\cos\frac{A}{2}\sin^2\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}$$

$$= \sqrt{\frac{4R^2}{r}}\sum\sin\frac{A}{2}$$

$$\Rightarrow \frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)} \overset{(b)}{\lesssim} \frac{2R}{\sqrt{r}}\sum\sin\frac{A}{2}$$

Now, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a)+bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as-s^2)$$

$$\Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2}$$
$$= as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_ar_a \Rightarrow n_a^2 \overset{(ii)}{=} s^2 - 2h_ar_a \text{ and analogs}$$

$$\therefore \sqrt{R}\sum\frac{n_a}{h_a} + \frac{r_a+r_b+r_c}{m_a+m_b+m_c}\sum\sqrt{2(r_a-r)} \overset{\text{via (b)}}{\lesssim} \sqrt{R}\sum\frac{n_a}{h_a} + \frac{2R}{\sqrt{r}}\sum\sin\frac{A}{2} = \sqrt{R}\sum\left(\frac{n_a}{h_a} + 2\sqrt{\frac{R}{r}}\sin\frac{A}{2}\right)$$

$$\overset{\text{via CBS and (ii) and analogs}}{\lesssim} \sqrt{2R}\sum\sqrt{\frac{s^2-2h_ar_a}{h_a^2} + \frac{4R}{r}\sin^2\frac{A}{2}}$$

$$= \sqrt{2R}\sqrt{\frac{s^2}{h_a^2} - \frac{8R\cos\frac{A}{2}\sin\frac{A}{2}s\tan\frac{A}{2}}{2rs} + \frac{4R}{r}\sin^2\frac{A}{2}}$$

$$=\sqrt{2R}\sum\sqrt{\frac{s^2}{h_a^2}-\frac{4R}{r}\sin^2\frac{A}{2}+\frac{4R}{r}\sin^2\frac{A}{2}}=s\sqrt{2R}\sum\frac{1}{h_a}$$

$$=\frac{s\sqrt{2R}}{r}\overset{\text{via }(a)}{\leq}\sum\frac{a}{\sqrt{h_a-2r}}\ \text{(QED)}$$

2447. In ΔABC the following relationship holds:

$$\frac{9}{4(n+1)(2R^2+r^2)}\leq\sum\frac{1}{a^2+nb^2}\leq\frac{1}{4(n+1)r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have:\sum\frac{1}{a^2+nb^2}\overset{CBS}{\geq}\frac{9}{\sum(a^2+nb^2)}=\frac{9}{(n+1)\sum a^2}=\frac{9}{2(n+1)(s^2-r^2-4Rr)}$$

$$\geq$$

$$\overset{Gerretsen}{\geq}\frac{9}{2(n+1)\left((4R^2+4Rr+3r^2)-r^2-4Rr\right)}=\frac{9}{4(n+1)(2R^2+r^2)}$$

$$Now,\sum\frac{1}{a^2+nb^2}\overset{CBS}{\leq}\sum\frac{1}{(1+n)^2}\left(\frac{1}{a^2}+n.\frac{1}{b^2}\right)=\frac{1}{n+1}\sum\frac{1}{a^2}\overset{Goldstone}{\leq}\frac{1}{n+1}.\frac{1}{4r^2}$$

$$=\frac{1}{4(n+1)r^2}$$

$$Therefore,\quad\frac{9}{4(n+1)(2R^2+r^2)}\leq\sum\frac{1}{a^2+nb^2}\leq\frac{1}{4(n+1)r^2}$$

2448. In ΔABC the following relationship holds:

$$\sum\frac{m_a}{h_a}\geq\frac{1}{4}\left(\sum\frac{b+c}{a}+\sum\frac{m_b+m_c}{m_a}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ know\ that:$$

$$m_a\overset{Tereshin}{\geq}\frac{b^2+c^2}{4R}=\frac{1}{2}.\frac{bc}{2R}.\frac{b^2+c^2}{bc}=\frac{h_a}{2}\left(\frac{b}{c}+\frac{c}{b}\right)\leftrightarrow\frac{b}{c}+\frac{c}{b}\leq\frac{2m_a}{h_a}\quad(1)$$

$$m_a, m_b, m_c \text{ can be length sides of triangle with}$$

$$: F_m = \frac{3}{4}F, \overline{m_a} = \frac{3}{4}a, \overline{h_a} = \frac{2F_m}{m_a} = \frac{3F}{2m_a}$$

$$In\ \Delta m_a m_b m_c, (1) \leftrightarrow$$

$$\frac{m_b}{m_c} + \frac{m_c}{m_b} \le \frac{2\overline{m_a}}{\overline{h_a}} \leftrightarrow \frac{m_b}{m_c} + \frac{m_c}{m_b} \le \frac{3}{2}a.\frac{2m_a}{3F} = \frac{2m_a}{h_a} \to \frac{m_b}{m_c} + \frac{m_c}{m_b} \le \frac{2m_a}{h_a} (2)$$

$$(1),(2) \to \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \le \frac{4m_a}{h_a} \ \ (and\ analogs)$$

$$\to \sum\left(\frac{b}{c} + \frac{c}{b}\right) + \sum\left(\frac{m_b}{m_c} + \frac{m_c}{m_b}\right) \le 4\sum\frac{m_a}{h_a} \leftrightarrow$$

$$\sum\left(\frac{b}{a} + \frac{c}{a}\right) + \sum\left(\frac{m_b}{m_a} + \frac{m_c}{m_a}\right) \le 4\sum\frac{m_a}{h_a}$$

$$Therefore, \quad \sum\frac{m_a}{h_a} \ge \frac{1}{4}\left(\sum\frac{b+c}{a} + \sum\frac{m_b + m_c}{m_a}\right)$$

2449. In $\Delta ABC, I$ −incenter, AK, BL, CM −diameters , the following relationship holds:

Prove that: $F_{\Delta KLM} \ge F_{\Delta ABC}$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by Marian Ursărescu-Romania

$$\mu(\widehat{ABK}) = 2C + A, \mu(\widehat{LAM}) = B + C$$

In ΔACK: $AK = 2R\sin\left(\frac{2C+A}{2}\right)$

In ΔAML: $LM = 2R\sin\frac{B+C}{2} = 2R\cos\frac{A}{2}$, then:

$$F_{\Delta KLM} = \frac{KL \cdot KM \cdot \sin(LKM)}{2} = \frac{1}{2} \cdot 2R\cos\frac{C}{2} \cdot 2R\cos\frac{B}{2} \cdot \cos\frac{A}{2}$$

Hence,

$$F_{\Delta KLM} = 2R^2\cos\frac{A}{2} \cdot \cos\frac{B}{2}\cos\frac{C}{2}; (1)$$

But: $F_{\Delta ABC} = 2R^2\sin A\sin B\sin C; (2)$.

From $(1),(2)$ it follows that:

$$\frac{F_{\Delta KLM}}{F_{\Delta ABC}}=\frac{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{\sin A\sin B\sin C}=\frac{1}{8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}};(3)$$

But in any ΔABC, we have: $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8};(4)$

From $(3),(4)$ it follows that:

$$\frac{F_{\Delta KLM}}{F_{\Delta ABC}}\ge 1$$

Therefore,

$$F_{\Delta KLM}\ge F_{\Delta ABC}$$

2450. If ABC is a nonisoscelles triangle, then prove that :

$$\sum\frac{a^4}{(a-b)(a-c)}>8\sqrt{3}S.$$

Proposed by D.M.Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$we\ have:\ \frac{a^4}{(a-b)(a-c)}=a^4\left(-\frac{1}{(b-a)(b-c)}-\frac{1}{(c-a)(c-b)}\right)$$

$$\to\sum\frac{a^4}{(a-b)(a-c)}=\frac{b^4-a^4}{(b-a)(b-c)}+\frac{c^4-a^4}{(c-a)(c-b)}=$$

$$=\frac{(b-a)(b+a)(b^2+a^2)}{(b-a)(b-c)}+\frac{(c-a)(c+a)(c^2+a^2)}{(c-a)(c-b)}$$

$$=\frac{(b+a)(b^2+a^2)-(c+a)(c^2+a^2)}{b-c}=$$

$$=\frac{b^3-c^3+a^2b-a^2c+ab^2-ac^2}{b-c}=\frac{(b-c)(a^2+b^2+c^2+ab+bc+ca)}{b-c}$$

$$=\sum a^2+\sum ab=$$

$$=\frac{1}{2}\sum(a+b)^2\overset{CBS}{\gtrsim}\frac{1}{2.3}\left[\sum(a+b)\right]^2=\frac{8s^2}{3}\overset{Mitrinovic}{\gtrsim}\frac{8.3\sqrt{3}sr}{3}=8\sqrt{3}S.$$

$$Therefore,\quad\sum\frac{a^4}{(a-b)(a-c)}>8\sqrt{3}S$$

2451. In ΔABC the following relationship holds:

$$6\left(\frac{r}{4R}\right)^{\frac{2}{3}} \le \sum \frac{s_a}{s_a+m_a} \le \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$We\ have: \sum \frac{s_a}{s_a+m_a} = \sum \frac{1}{1+\frac{m_a}{s_a}} \overset{s_a \le m_a}{\le} \sum \frac{1}{1+1} = \frac{3}{2}.$$

$$\sum \frac{s_a}{s_a+m_a} = \sum \frac{1}{1+\frac{m_a}{s_a}} = \sum \frac{1}{1+\frac{b^2+c^2}{2bc}} = \sum \frac{2bc}{(b+c)^2} \overset{AM-GM}{\ge} 6\left(\prod \frac{bc}{(b+c)^2}\right)^{\frac{1}{3}} =$$

$$= 6\left(\frac{abc}{\prod(a+b)}\right)^{\frac{2}{3}} \overset{AM-GM}{\ge} 6\left(\frac{27abc}{(\sum(a+b))^3}\right)^{\frac{2}{3}} = 6\left(\frac{27.4sRr}{64s^3}\right)^{\frac{2}{3}} = 6\left(\frac{27Rr}{4(2s)^2}\right)^{\frac{2}{3}} \ge$$

$$\overset{Mitrinovic}{\ge} 6\left(\frac{27Rr}{4\left(3\sqrt{3}R\right)^2}\right)^{\frac{2}{3}} = 6\left(\frac{r}{4R}\right)^{\frac{2}{3}}$$

$$Therefore, \quad 6\left(\frac{r}{4R}\right)^{\frac{2}{3}} \le \sum \frac{s_a}{s_a+m_a} \le \frac{3}{2}$$

2452. In $\Delta ABC, n_a$ —Nagel' s cevian, g_a —Gergonne' s cevian, the following relationship holds :

$$\sqrt{\prod\left(\frac{{n_a}^2+{g_a}^2}{{h_b}^2+{h_c}^2}\right)}.\sum \frac{w_b+w_c}{a} \ge \frac{1}{2}\sum\left(\frac{m_a w_a}{g_a r_a}+\frac{2h_a}{n_a+s}\right).\left(\sqrt{\frac{r_a+r_c}{r_a+r_b}}+\sqrt{\frac{r_a+r_b}{r_a+r_c}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$Let\ us\ prove: \ m_a w_a \le n_a g_a.$$

$$We\ known\ that: \ {n_a}^2 = s(s-a)+\frac{(b-c)^2}{a}s, \qquad {g_a}^2 = s(s-a)-\frac{(b-c)^2}{a}(s-a)$$

$$4[(n_a g_a)^2-(m_a w_a)^2] =$$

$$= s(s-a)\left[2(s-a)+\frac{2(b-c)^2}{a}\right]\left[2s-\frac{2(b-c)^2}{a}\right]-4\frac{(b+c)^2+(b-c)^2-a^2}{4}.\frac{4bcs(s-a)}{(b+c)^2}$$

$$= s(s-a)\left[(b+c)^2-a^2+4(b-c)^2-\frac{4(b-c)^4}{a^2}-4bc+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$= s(s-a)\left[(b+c)^2-4bc-a^2+\frac{4(b-c)^2(a^2-(b-c)^2)}{a^2}+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$= s(s-a)\left[(b-c)^2-a^2+\frac{4(b-c)^2(a^2-(b-c)^2)}{a^2}+4bc.\frac{a^2-(b-c)^2}{(b+c)^2}\right]$$

$$= s(s-a)[a^2-(b-c)^2]\left(-1+\frac{4(b-c)^2}{a^2}+\frac{4bc}{(b+c)^2}\right)=4F^2\left(\frac{4(b-c)^2}{a^2}-\frac{(b-c)^2}{(b+c)^2}\right)$$

$$= 4F^2(b-c)^2\frac{(2b+2c+a)(2b+2c-a)}{a^2(b+c)^2}\geq 0 \;\rightarrow\; m_a w_a \leq n_a g_a \;\rightarrow$$

$$\frac{m_a w_a}{g_a}\leq n_a \;(and\ analogs)$$

$$\rightarrow \sum\left(\frac{m_a w_a}{g_a r_a}+\frac{2h_a}{n_a+s}\right)\leq \sum\left(\frac{n_a}{r_a}+\frac{2h_a(s-n_a)}{s^2-{n_a}^2}\right)\overset{{n_a}^2=s^2-2r_a h_a}{\cong}\sum\left(\frac{n_a}{r_a}+\frac{2h_a(s-n_a)}{2r_a h_a}\right)$$

$$= \sum\left(\frac{n_a}{r_a}+\frac{s-n_a}{r_a}\right)=\sum\frac{s}{r_a}=\frac{s}{r}\;\rightarrow\;\sum\left(\frac{m_a w_a}{g_a r_a}+\frac{2h_a}{n_a+s}\right)\leq\frac{s}{r}\quad(1)$$

$$We\ have:\ \sum\frac{w_b+w_c}{a}=\sum w_a\left(\frac{1}{b}+\frac{1}{c}\right)=\sum\frac{2bc}{b+c}.\cos\frac{A}{2}.\frac{b+c}{bc}=2\sum\cos\frac{A}{2}$$

$$\rightarrow \sum\frac{w_b+w_c}{a}=2\sum\cos\frac{A}{2}\quad(2)$$

$$Now,\quad we\ known\ that:\ {n_a}^2=s(s-a)+\frac{(b-c)^2}{a}s,$$

$${g_a}^2=s(s-a)-\frac{(b-c)^2}{a}(s-a)$$

$$\rightarrow {n_a}^2+{g_a}^2=2s(s-a)+(b-c)^2\geq 2s(s-a)\;(i)$$

and

$$2({n_a}^2+{g_a}^2)=(a+b+c)(-a+b+c)+2(b-c)^2=(b+c)^2-a^2+2(b-c)^2=$$

$$=2(b^2+c^2)+(b-c)^2-a^2=2(b^2+c^2)-4(s-b)(s-c)\;\rightarrow\;b^2+c^2$$

$$={n_a}^2+{g_a}^2+2(s-b)(s-c)$$

$$\to \frac{n_a^2+g_a^2}{b^2+c^2}=\frac{n_a^2+g_a^2}{n_a^2+g_a^2+2(s-b)(s-c)}$$

$$=\frac{1}{1+\frac{2(s-b)(s-c)}{n_a^2+g_a^2}}\overset{(i)}{\geq}\frac{1}{1+\frac{2(s-b)(s-c)}{2s(s-a)}}=\frac{1}{1+\tan^2\frac{A}{2}}$$

$$\to \frac{n_a^2+g_a^2}{b^2+c^2}\geq \cos^2\frac{A}{2}\to \frac{n_a^2+g_a^2}{h_b^2+h_c^2}=\frac{(bc)^2}{(2sr)^2}.\frac{n_a^2+g_a^2}{b^2+c^2}\geq\frac{4R^2}{a^2}.\cos^2\frac{A}{2}$$

$$\to \prod\left(\frac{n_a^2+g_a^2}{h_b^2+h_c^2}\right)\geq\prod\left(\frac{4R^2}{a^2}.\cos^2\frac{A}{2}\right)=\frac{(4R^2)^3}{(4sRr)^2}.\left(\frac{s}{4R}\right)^2=\frac{R^2}{4r^2}\to\prod\left(\frac{n_a^2+g_a^2}{h_b^2+h_c^2}\right)\geq\left(\frac{R}{2r}\right)^2 \quad (3)$$

$$We\ also\ know\ that:\ r_b+r_c=4R\cos^2\frac{A}{2}\to\sqrt{\frac{r_a+r_c}{r_a+r_b}}+\sqrt{\frac{r_a+r_b}{r_a+r_c}}$$

$$=\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \quad (4)$$

From (1), (2), (3), (4) *it is suffices to prove*

$$:\ \frac{R}{2r}.2\sum\cos\frac{A}{2}\overset{(*)}{\geq}\frac{1}{2}.\frac{s}{r}\left(\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\right)$$

$$\leftrightarrow\sum\cos\frac{A}{2}\geq\frac{s}{2R}\left(\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\right)=2\left(\prod\cos\frac{A}{2}\right).\left(\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\right)$$

$$\leftrightarrow\sum\cos\frac{\pi-2A}{2}\geq2\left(\prod\cos\frac{\pi-2A}{2}\right).\left(\frac{\cos\frac{\pi-2B}{2}}{\cos\frac{\pi-2C}{2}}+\frac{\cos\frac{\pi-2C}{2}}{\cos\frac{\pi-2B}{2}}\right)$$

for any triangle with angles $\pi-2A,\pi-2B,\pi-2C.$

$$\leftrightarrow\sum\sin A\geq2\left(\prod\sin A\right).\left(\frac{\sin B}{\sin C}+\frac{\sin C}{\sin B}\right)\leftrightarrow\frac{s}{R}\geq2.\frac{sr}{2R^2}.\left(\frac{b}{c}+\frac{c}{b}\right)$$

$$\leftrightarrow\frac{b}{c}+\frac{c}{b}\leq\frac{R}{r}\ (Viorel\ Bandila\ inequality)\to(*)\ is\ true.$$

Therefore,

$$\sqrt{\prod\left(\frac{n_a^2+g_a^2}{h_b^2+h_c^2}\right)}.\sum\frac{w_b+w_c}{a}\geq\frac{1}{2}\sum\left(\frac{m_aw_a}{g_ar_a}+\frac{2h_a}{n_a+s}\right).\left(\sqrt{\frac{r_a+r_c}{r_a+r_b}}+\sqrt{\frac{r_a+r_b}{r_a+r_c}}\right)$$