POPOVICIU'S INEQUALITY REVISITED

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Abstract. In this paper we present some new inequalities using famous Popoviciu's inequality.

Theorem 1.
If a, b, c are positive real numbers, then $a+b+c+3\cdot\sqrt[3]{abc} \geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})$

Proof. It is well-known Popoviciu's inequality:
If $f : I \to \mathbb{R}, I \subset \mathbb{R}$ is an interval and f is a continue and convex function on I, then:

$$\frac{f(x)+f(y)+f(z)}{3}+f\Big(\frac{x+y+z}{3}\Big) \geq \frac{2}{3}\Big(f\Big(\frac{x+y}{2}\Big)+f\Big(\frac{y+z}{2}\Big)+f\Big(\frac{z+x}{2}\Big)\Big), \forall x, y, z \in I$$

For $I = \mathbb{R}$ and $f(x) = e^x$, then $f'(x) = f''(x) = e^x > 0, \forall x \in \mathbb{R}$, so we can apply Popoviciu's inequality.
Then:

$$\frac{e^x+e^y+e^z}{3}+e^{\frac{x+y+z}{3}} \geq \frac{2}{3}\Big(e^{\frac{x+y}{2}}+e^{\frac{y+z}{2}}+e^{\frac{z+x}{2}}\Big)$$

where we put $x = \ln a, y = \ln b, z = \ln c$, and we obtain the desired inequality. $\square$

Theorem 2.
If $a, b, c \in \mathbb{R}_+^*$, then:

$$a^2+b^2+c^2+3\cdot\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)$$

Proof. If $I = \mathbb{R}$ and $f(x) = e^{2x}$, then by Popoviciu's inequality $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x} > 0, \forall x \in \mathbb{R}_+$, we have:

$$\frac{e^{2x}+e^{2y}+e^{2z}}{3}+e^{\frac{2(x+y+z)}{3}} \geq \frac{2}{3}\Big(e^{\frac{2(x+y)}{2}}+e^{\frac{2(y+z)}{2}}+e^{\frac{2(z+x)}{2}}\Big)$$

where we take $x = \ln a, y = \ln b, z = \ln c$ $\square$

Theorem 3.
If $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a convex function on $\mathbb{R}_+^*$, then:

$$3(f^2(x)+f^2(y)+f^2(z))-9f^2\Big(\frac{x+y+z}{3}\Big) \geq (f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2$$

Proof. We have $f''(x) > 0, \forall x \in \mathbb{R}_+^*$. We consider the function $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$, $g = f^2$. So, $g'(x) = 2f(x)f'(x), \forall x \in \mathbb{R}_+^*$ and
$g''(x) = 2(f'(x))^2+2f(x)f''(x) > 0, \forall x \in \mathbb{R}_+^*$.
So, g is convex on $\mathbb{R}_+^*$. By Jensen's inequality we have:

$$f(x)+f(y)+f(z) \geq 3f\Big(\frac{x+y+z}{3}\Big), \forall x, y, z \in \mathbb{R}_+^* \tag{1}$$

and

(2) $g(x)+g(y)+g(z) \geq 3g\left(\frac{x+y+z}{3}\right) \Leftrightarrow f^2(x)+f^2(y)+f^2(z) \geq 3f^2\left(\frac{x+y+z}{3}\right)$

By (1) we deduce that:

(3)

$$(f(x)+f(y)+f(z))^2 \geq 9f^2\left(\frac{x+y+z}{3}\right) \Leftrightarrow -3f^2\left(\frac{x+y+z}{3}\right) \geq -\frac{(f(x)+f(y)+f(z))^2}{3}$$

By (2) and (3) we deduce that:

$$f^2(x)+f^2(y)+f^2(z)-3f^2\left(\frac{x+y+z}{3}\right) \geq f^2(x)+f^2(y)+f^2(z)-\frac{(f(x)+f(y)+f(z))^2}{3} =$$

$$= \frac{1}{3}[(f(x)-f(y))^2+(f(y)-f(z))^2+(f(z)-f(x))^2]$$

and we are done. □

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