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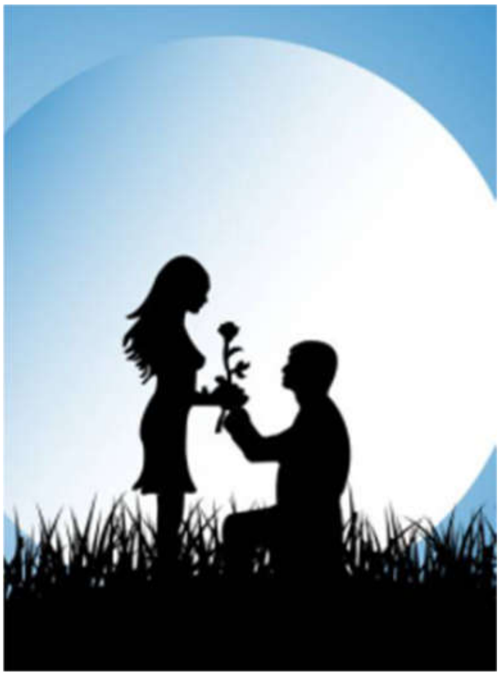

**If $a,b,c>0$ such that $a^2+b^2+c^2=3$ and $\lambda\ge 0$ then:**

$$\frac{a}{a+\lambda bc}+\frac{b}{b+\lambda ca}+\frac{c}{c+\lambda ab}\le\frac{3}{1+\lambda abc}$$

***Proposed by Marin Chirciu-Romania***

***Solution 1 by George Florin Șerban-Romania, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco***

***Solution 1 by George Florin Șerban-Romania***

$$\sum_{cyc}\frac{a}{a+\lambda bc}=\sum_{cyc}\left(1-\frac{\lambda bc}{a+\lambda bc}\right)=3-\lambda\sum_{cyc}\frac{bc}{a+\lambda bc}\overset{(*)}{\le}\frac{3}{1+\lambda abc}$$

$$(*)\Leftrightarrow\lambda\sum_{cyc}\frac{bc}{a+\lambda bc}\ge 3-\frac{3}{1+\lambda abc}=\frac{3\lambda abc}{1+\lambda abc}$$

**If $\lambda=0\Rightarrow\sum\frac{a}{a}=\sum 1=3\ge\frac{3}{1}$ true.**

**If $\lambda>0$ then**

$$\frac{1}{abc}\sum_{cyc}\frac{bc}{a+\lambda bc}\overset{(?)}{\ge}\frac{3}{1+\lambda abc}$$

$$\Rightarrow \sum_{cyc} \frac{1}{a(a+\lambda bc)} \geq \frac{1}{1+\lambda abc}$$

$$\Rightarrow \sum_{cyc} \frac{1}{a^2+\lambda abc} \overset{Bergstrom}{\geq} \frac{9}{\sum a^2 + \lambda \sum abc} = \frac{9}{3+3\lambda abc} = \frac{1}{1+\lambda abc}$$

**Therefore,**

$$\frac{a}{a+\lambda bc} + \frac{b}{b+\lambda ca} + \frac{c}{c+\lambda ab} \leq \frac{3}{1+\lambda abc}$$

***Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco***

$$Let\ r = \lambda abc \geq 0\ and\ f(x) = \frac{x}{x+r}, a > 0.$$

$$We\ have:\ f'(x) = \frac{r}{(x+r)^2}\ and\ f''(x) = -\frac{2r}{(x+r)^3} \leq 0$$

$$\rightarrow \sum \frac{a}{a+\lambda bc} = \sum \frac{a^2}{a^2+r} = \sum f(a^2) \overset{Jensen}{\leq} 3f\left(\frac{1}{3}\sum a^2\right) =$$

$$= 3f(1) = \frac{3}{1+r} = \frac{3}{1+\lambda abc} \quad \rightarrow \quad \sum \frac{a}{a+\lambda bc} \leq \frac{3}{1+\lambda abc}.$$

□