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**Find $z_1, z_2, z_3 \in C, Rez_i < 0, i = \overline{1,3}$, such that $\exists a, b, c > 0$ :**

$$\sum_{cyc} |a - z_i|^2 \le 2\sum_{cyc} a|z_i|$$

***Proposed by Dan Radu Seclăman-Romania***

***Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Ravi Prakash-New Delhi-India***

***Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco***

$$(*)\ \sum_{cyc} |a - z_i|^2 \le 2\sum_{cyc} a|z_i|$$

$$Let\ Rez_i = -x_i < 0, Imz_i = y_i, i = \overline{1,3}.$$

$$(*) \leftrightarrow \sum_{cyc} [(a + x_i)^2 + y_i^2] \le 2\sum_{cyc} a\sqrt{x_i^2 + y_i^2}$$

$$\leftrightarrow \sum_{cyc} [a^2 + (x_i^2 + y_i^2)] + 2\sum_{cyc} ax_i \le 2\sum_{cyc} a\sqrt{x_i^2 + y_i^2}$$

$$\leftrightarrow \sum_{cyc} \left(a - \sqrt{x_i^2 + y_i^2}\right)^2 + 2\sum_{cyc} ax_i \le 0$$

$$\textbf{But we have}: a,b,c,x_i > 0, i=\overline{1,3} \rightarrow \sum_{cyc}\left(a-\sqrt{x_i^2+y_i^2}\right)^2 \ge 0 \text{ and } 2\sum_{cyc} ax_i > 0.$$

$$\rightarrow 0 < \sum_{cyc}\left(a-\sqrt{x_i^2+y_i^2}\right)^2 + 2\sum_{cyc} ax_i \le 0.$$

$$\text{So they don't exist } z_1,z_2,z_3 \in C \text{ verifying } (*).$$

***Solution 2 by Ravi Prakash-New Delhi-India***

**We know that:**

$$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2Re(z_1\overline{z_2}).$$

**Now,** $|a-z_1|^2+|b-z_2|^2+|c-z_3|^2-2(a|z_1|+b|z_2|+c|z_3|) =$

$$= a^2+b^2+c^2+|z_1|^2+|z_2|^2+|z_3|^2-2a(Re(z_1)+|z_1|)-2b(Re(z_2)+|z_2|)-$$

$$-2c(Re(z_3)+|z_3|) =$$

$$= (a-|z_1|)^2+(b-|z_2|)^2+(c-|z_3|)^2-2aRe(z_1)-2bRe(z_2)-2cRe(z_3) > 0$$

$$\because a,b,c > 0, Re(z_1), Re(z_2), Re(z_3) < 0)$$

**Thus, no such** $a,b,c$ **exist.**