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ABOUT LEHMUS-PADOA'S INEQUALITY

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*In this note are presented and proved some generalizations and refinements for a known inequality in the geometry of triangle (known until recently as **Padoa's inequality**, but which according to a very interesting and learned article from* [3] – *it was formulated long before, by **Lehmus*** [6]) *.Several applications of **Lehmus-Padoa inequality** refinement are also presented .*

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In triangle's geometry , the following very beautiful inequality is known (v. [1] ,[3]-[6] , [8]-[13]) , which impresses especially by its simplicity, circularity, homogeneity and symmetry :

1.*Theorem* (*Lehmus – Padoa's inequality*)

In any triangle ABC , of sides a, b, c occurs the inequality ,

$$(-a+b+c)(a-b+c)(a+b-c) \leq abc; \ (1)$$

equality taking place in the case of the equilateral triangle.

Brief historical note: In the literature , this inequality is commonly known as *Padoa's inequality*, it being stated and demonstrated in the work [13] of 1925. But in 1985, the well-known geometer *H.S.M. Coxeter* in [3], consulting the very rare book [6] from 1820 of *D.C.L. Lehmus*, finds the inequality explicitly presented and demonstrated on pages 27-28. For this reason he calls the inequality (1) the *inequality of Lehmus*.

In what follows, by aggregating the two attributions, and as a necessary historical rehabilitation, we will call it the inequality of *Lehmus - Padoa*.The notoriety of relation (1) also lies in the fact that this inequality is equivalent to many inequalities - algebraic and geometric - no less famous. Thus, the inequality (1) is equivalent to *Euler's* famous triangle inequality, $R \geq 2r; (2)$

as demonstrated by *Richard Balzer*, as early as 1870, a demonstration resumed in [3]. See also [1] and [12].

Also, in [12] and [1], is demonstrated the equivalence with the well-known elementary *inequality of Cesaro*, from 1880, $(x+y)(y+z)(z+x) \geq 8xyz, (x, y, z \geq 0); (3)$

Through developments and arrangements in (1), are obtained - equivalent, the following ***Schur-type*** inequalities :

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a); \text{(AUO 1975)}; (4)$$

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b) \geq 0; (5)$$

$$a^2(b+c-a)+b^2(c+b-a)+c^2(a+b-c) \leq 3abc; \text{(IMO 1964)}$$

(*see* Engel – Problem solving , p.167 **)**

As we can see, the inequality of *Lehmus* - *Padoa* - although elementary - through its special aesthetics it also attracted the attention of some redoubtable mathematicians .

This inequality has known several demonstrations (starting even with those of its protagonists ...) , extensions, generalizations as well as various geometric interpretations; *see* [3], [5], [12].

Below, we present a simple, very short proof - and it seems, genuine. More importantly - the method used also suggests concrete ways of generalization and refinement of the *Lehmus-Padoa* inequality , as will be seen further .

A Proof of Lehmus–Padoa Inequality

The expressions in parentheses on the left side of the *Lehmus-Padoa*'s inequality are positive, suggesting the use of *GM –AM inequality*. Indeed, with the next arrangement, we have ,

$$(-a+b+c)(a-b+c)(a+b-c)=$$

$$=\sqrt{(-a+b+c)(a-b+c)}\cdot\sqrt{(a-b+c)(a+b-c)}.\sqrt{(-a+b+c)(a+b-c)} \leq$$

$$\leq \frac{(-a+b+c)+(a-b+c)}{2}\cdot\frac{(a-b+c)(a+b-c)}{2}\cdot\frac{(-a+b+c)(a+b-c)}{c}=cab$$

Equality occurs if and only if $(-a+b+c)=(a-b+c)=(a+b-c) \Leftrightarrow a=b=c$.

The above demonstration immediately suggests the following generalization ,

2.*Proposition*_ (*a weighted generalization of Lehmus – Padoa's inequality*)

If $a,b,c>0$and $m,n,p\in\mathbb{R}$, distinct such that $\alpha a+\beta b+\gamma c\geq 0$, and where $\{\alpha,\beta,\gamma\}=\{m,n,p\}$, prove that ,

$$8(ma+nb+pc)(na+pb+mc)(pa+mb+nc) \leq$$

$$\leq [(m+n)a+(n+p)b+(p+m)c][(n+p)a+(p+m)b+(m+n)c][(p+m)a+(m+n)b+(n+p)c]$$

Proof.

The relations in the condition guarantee the non-negativity of the inequality factors, so again the possibility of using the *AM-GM inequality.*

With the same demonstration idea above, we have successively :

$$(ma+nb+pc)(na+pb+mc)(pa+mb+nc)=$$
$$=\sqrt{(ma+nb+pc)(na+pb+mc)}\cdot\sqrt{(na+pb+mc)(pa+mb+nc)}$$
$$\cdot\sqrt{(pa+mb+nc)(ma+nb+pc)}\le$$
$$\le\frac{(ma+nb+pc)+(na+pb+mc)}{2}+\frac{(na+pb+mc)+(pa+mb+nc)}{2}+\frac{(pa+mb+nc)+(ma+nb+pc)}{2}$$
$$=\frac{1}{8}[(m+n)a+(n+p)b+(p+m)c][(n+p)a+(p+m)b+(m+n)c][(p+m)a+(m+n)b+(n+p)c]$$

By customizing in (6), with the values : $m=n=1$and $p=-1$(or variants) , the *Lehmus –Padoa*'s inequality is obtained. Returning to the initial inequality, with the notations

$G(u,v)=\sqrt{uv}, A(u,v)=\frac{u+v}{2}$, for binary means:

geometric and arithmetic, the proof of the inequality *Lehmus - Padoa* - is transcribed in the language of means - in the form :

$$(-a+b+c)(a-b+c)(a+b-c)=$$
$$=G(-a+b+c,a-b+c)\cdot G(a-b+c,a+b-c)\cdot G(-a+b+c,a+b-c)\le$$
$$\le A(-a+b+c,a-b+c)\cdot A(a-b+c,a+b-c)\cdot A(-a+b+c,a+b-c)=cab;(7)$$

This specific arrangement also suggests the next more general property of refining inequality (1) , a procedure that can be used practically in refining any inequality that uses - in the same manner – *inequality of means* in its proof .

3. *Theorem* (*A refinement of Lehmus–Padoa Inequality*)

If M is a binary mean that refines the inequality between the geometric mean and the arithmetic mean (i.e.$G(u,v)\le M(u,v)\le A(u,v), \forall u,v>0$ then we get refinement for inequality (1) ,

$$(-a+b+c)(a-b+c)(a+b-c)\le$$
$$\le M(-a+b+c,a-b+c)\cdot M(a-b+c,a+b-c)\cdot M(-a+b+c,a+b-c)\le abc;(8)$$

for any a,b,c , the sides of a triangle.

Proof The demonstration results immediately by inserting in (7) the mean M - which refines the inequality GM–AM. For a more general case it can be tracked to the end of these notes.

By customizing the mean M , numerous (theoretically, infinitely many) refinements of *Lehmus-Padoa* *inequality* can be obtained.

4. *Application*

If we take $M(u,v)=\frac{A(u,v)+G(u,v)}{2}$, for which we obviously have ,$G(u,v)\le M(u,v)\le A(u,v)$,

How,

$$M(-a+b+c,a-b+c)=\frac{\frac{(-a+b+c)+(a-b+c)}{2}+\sqrt{(-a+b+c)(a-b+c)}}{2}$$

$$(-a+b+c)(a-b+c)(a+b-c)\le$$

$$\le\frac{c+\sqrt{(-a+b+c)(a-b+c)}}{2}\cdot\frac{a+\sqrt{(a-b+c)(a+b-c)}}{2}\cdot\frac{b+\sqrt{(-a+b+c)(a+b-c)}}{2}$$

$$\le abc;(9)$$

5. *Application*

If we take $M(u,v)=\sqrt{G(u,v)\cdot A(u,v)}$, for which we obviously have, $G(u,v)\le M(u,v)\le A(u,v)$

How, $M(-a+b+c,a-b+c)=\sqrt{\sqrt{(-a+b+c)(a-b+c)}\cdot\frac{(-a+b+c)(a-b+c)}{2}}=$

$=\sqrt[4]{(-a+b+c)(a-b+c)c^2}$ and analogues, with (8) result refinement,

$$(-a+b+c)(a-b+c)(a+b-c)\le\sqrt{abc(-a+b+c)(a-b+c)(a+b-c)};(10)$$

(Moreover, this inequality results directly from the *Lehmus – Padoa*'s inequality, through its application for each of the two component inequalities). If we also take into account the inequality

$\sqrt{G(u,v)\cdot A(u,v)}\le\frac{G(u,v)+A(u,v)}{2}$, from (7) and (8) we obtain double refinement ,

$$(-a+b+c)(a-b+c)(a+b-c)\le\sqrt{abc(-a+b+c)(a-b+c)(a+b-c)}\le$$

$$\le\frac{c+\sqrt{(-a+b+c)(a-b+c)}}{2}\cdot\frac{a+\sqrt{(a-b+c)(a+b-c)}}{2}\cdot\frac{b+\sqrt{(-a+b+c)(a+b-c)}}{2}\le$$

$$\le abc;(11)$$

6. *Remark* In the same way we can use in *Theorem 3*, instead of the mean M, the following *trigonometric mean*,

which refines - even continuously - the inequality of means , see [7],

$$T(u,v)=\sqrt{(u\sin^2x+v\cos^2x)(u\sin^2x+v\cos^2x)},x\in\left[0,\frac{\pi}{2}\right];(12)$$

or its algebraic variant, $U(u,v)=\sqrt{(pu+qv)(qu+pv)},p,q>0,p+q=1;(13)$

7. *Application*

Two transcendent refinements of the classical inequality of two-variable means are also well known in the literature, $G(u,v) \le L(u,v) \le I(u,v) \le A(u,v),\ u,v>0$, where : $L(u,v)=\frac{u-v}{\log u-\log v}$ is the ***logarithmic mean*** - introduced by *Polya and Szegö* in 1951, [14] and imposed by *B.C. Carlson* in [2] , respectively , $I(u,v)=\frac{1}{e}\cdot\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}}$ which is the ***identric mean*** , introduced and studied by *K. B. Stolarsky* in [15] . Based on the above inequality chain and in the spirit of *Theorem 3*, we have the multiple refinement of the *Lehmus - Padoa inequality* :

$$(-a+b+c)(a-b+c)(a+b-c)=$$
$$G(-a+b+c,a-b+c)\cdot G(a-b+c,a+b-c)\cdot G(-a+b+c,a+b-c)\le$$
$$\le L(-a+b+c,a-b+c)\cdot L(a-b+c,a+b-c)\cdot L(-a+b+c,a+b-c)\le$$
$$\le I(-a+b+c,a-b+c)\cdot I(a-b+c,a+b-c)\cdot I(-a+b+c,a+b-c)\le$$
$$\le A(-a+b+c,a-b+c)\cdot A(a-b+c,a+b-c)\cdot A(-a+b+c,a+b-c)=abc;(14)$$

A generalization of the *Lehmus – Padoa* inequality for $n-$ variables was presented in the statement of a problem proposed by *Mitrinović* and solved and generalized by *Adamović*. See : [5], [9] ,[10] .

8*. Theorem* (*the inequality of Mitrinović – Adamović*)

If $a_k>0, k\in\{1,2\dots,n\}$and $S_n=a_1+a_2+\cdots+a_n$, then occurs the inequality ,

$$[S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot\ldots\cdot[S-(n-1)a_n]\le a_1a_2\dots a_n;(15)$$

where all parentheses in the left side are positive. The ***Proof*** can be extracted from that of the refined inequality of inequality (16) - which follows. A refinement similar to that of *Theorem 3* is given by the following ,

9.*Theorem*_ (*refinement of Mitrinović – Adamović's inequality*)

If M_k is an means that refines the inequality between the geometric mean and the arithmetic mean of the numbers $a_1,a_2,\dots,a_k$, (meaning $G_k(a_1,a_2,\dots,a_k)\le M_k(a_1,a_2,\dots,a_k)\le A_k(a_1,a_2,\dots,a_k)$ $\forall a_1,a_2,\dots,a_k>0$ then the next refinements of inequality (15) takes place ,

$$[S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot\ldots\cdot[S-(n-1)a_n]\le$$
$$\le M_{n-1}(S-(n-1)a_1,S-(n-1)a_2,\dots,S-(n-1)a_{n-1})\cdot$$

$$\cdot M_{n-1}(S-(n-1)a_2, S-(n-1)a_3, \dots, S-(n-1)a_n)\cdot \dots \cdot$$
$$\cdot M_{n-1}(S-(n-1)a_n, S-(n-1)a_1, \dots, S-(n-1)a_{n-2}) \le a_1a_2 \dots a_n; (16)$$

The ***Proof*** is similar to that of *Theorem* 1 and consists in applying the inequality between the geometric mean and the arithmetic mean for $n-1$ variables ,

$$[S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot \dots \cdot[S-(n-1)a_n]=$$
$$=\sqrt[n]{[S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot \dots \cdot[S-(n-1)a_n]}\cdot$$
$$\cdot\sqrt[n]{[S_n+(n-1)a_2]\cdot[S-(n-1)a_3]\cdot \dots \cdot[S-(n-1)a_{n-1}]}\cdot \dots \cdot$$
$$\cdot\sqrt[n]{[S_n+(n-1)a_n]\cdot[S-(n-1)a_1]\cdot \dots \cdot[S-(n-1)a_{n-2}]}=$$
$$=G_{n-1}([S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot \dots \cdot[S-(n-1)a_n])\cdot$$
$$\cdot G_{n-1}([S_n+(n-1)a_2]\cdot[S-(n-1)a_3]\cdot \dots \cdot[S-(n-1)a_{n-1}])\cdot \dots \cdot$$
$$\cdot G_{n-1}([S_n+(n-1)a_n]\cdot[S-(n-1)a_1]\cdot \dots \cdot[S-(n-1)a_{n-2}])\le$$
$$\le M_{n-1}([S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot \dots \cdot[S-(n-1)a_n])\cdot$$
$$\cdot M_{n-1}([S_n+(n-1)a_2]\cdot[S-(n-1)a_3]\cdot \dots \cdot[S-(n-1)a_{n-1}])\cdot \dots \cdot$$
$$\cdot M_{n-1}([S_n+(n-1)a_n]\cdot[S-(n-1)a_1]\cdot \dots \cdot[S-(n-1)a_{n-2}])\le$$
$$\le A_{n-1}([S_n+(n-1)a_1]\cdot[S-(n-1)a_2]\cdot \dots \cdot[S-(n-1)a_n])\cdot$$
$$\cdot A_{n-1}([S_n+(n-1)a_2]\cdot[S-(n-1)a_3]\cdot \dots \cdot[S-(n-1)a_{n-1}])\cdot \dots \cdot$$
$$\cdot A_{n-1}([S_n+(n-1)a_n]\cdot[S-(n-1)a_1]\cdot \dots \cdot[S-(n-1)a_{n-2}])=$$
$$=\frac{[S_n+(n-1)a_1]+[S-(n-1)a_2]+\dots+[S-(n-1)a_n]}{n-1}+$$
$$+\frac{[S_n+(n-1)a_2]+[S-(n-1)a_3]+\dots+[S-(n-1)a_{n-1}]}{n-1}+\dots$$
$$+\frac{[S_n+(n-1)a_n]+[S-(n-1)a_3]+\dots+[S-(n-1)a_{n-2}]}{n-1}=$$
$$=\frac{(n-1)S-(n-1)(S-a_n)}{n-1}\cdot\frac{(n-1)S-(n-1)(S-a_1)}{n-1}\cdot \dots \cdot\frac{(n-1)S-(n-1)(S-a_{n-1})}{n-1}=$$
$$=a_1a_2 \dots a_n$$

Equality occurs if and only if $a_1=a_2=\dots=a_k$. By customizing M_k, with different specific means , similar to those in *Applications* 4 -7, numerous refinements of for inequality (15) can be obtained .

We can also formulate a generalization of *Mitrinović – Adamović inequality*, similar to that in *Theorem* 2, with the same type of proof as in *Theorem* 8, but which - more for typographical reasons - we will leave to the readers.

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