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ABOUT AN INEQUALITY BY MARIAN URSĂRESCU -XX

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1) In ΔABC the following relationship holds:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}\geq\frac{54R}{2\left(\frac{R}{r}\right)^2+1}$$

Proposed by Marian Ursărescu-Romania

Solution. **Lemma. In ΔABC the following relationship holds:**

$$\sum_{cyc}(b+c)\tan\frac{A}{2}=4(R+r)$$

Proof. Using $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ we get:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}=\sum_{cyc}(b+c)\sqrt{\frac{(s-b)(s-c)}{s(s-c)}}=\sum_{cyc}(b+c)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)}=$$

$$=\frac{F}{s}\sum_{cyc}\frac{b+c}{s-a}=r\cdot4\left(1+\frac{R}{r}\right)=4(R+r),\text{which follows from:}$$

$$\sum_{cyc}\frac{b+c}{s-a}=4\left(1+\frac{R}{r}\right)$$

Let's get back to the main problem. Using Lemma, inequality becomes as:

$$4(R+r)\geq\frac{54R}{2\left(\frac{R}{r}\right)^2+1}\Leftrightarrow2(R+r)\left[2\left(\frac{R}{r}\right)^2+1\right]\geq27R\Leftrightarrow$$

$$2(R+r)\cdot\frac{2R^2+r^2}{r^2}\geq27R\Leftrightarrow2(R+r)(2R^2+r^2)\geq27Rr^2\Leftrightarrow$$

$$2(2R^3+2R^2r+Rr^2+r^3)\geq27Rr^2\Leftrightarrow4R^3+4R^2r+2Rr^2+2r^3\geq27Rr^2\Leftrightarrow$$

$4R^3+4R^2r-25Rr^2+2r^3\geq0\Leftrightarrow(R-2r)(4R^2+12Rr-r^2)\geq0$, which is true from $R\geq2r\ (Euler)$.

2) In ΔABC the following relationship holds:

$$12r \le \sum_{cyc} (b+c)\tan\frac{A}{2} \le 6R$$

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Solution. **Lemma. In ΔABC the following relationship holds:**

$$\sum_{cyc} (b+c)\tan\frac{A}{2} = 4(R+r)$$

Proof. Using $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ we get:

$$\sum_{cyc}(b+c)\tan\frac{A}{2} = \sum_{cyc}(b+c)\sqrt{\frac{(s-b)(s-c)}{s(s-c)}} = \sum_{cyc}(b+c)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)} =$$

$$= \frac{F}{s}\sum_{cyc}\frac{b+c}{s-a} = r\cdot 4\left(1+\frac{R}{r}\right) = 4(R+r), \text{which follows from:}$$

$$\sum_{cyc}\frac{b+c}{s-a} = 4\left(1+\frac{R}{r}\right)$$

Using Lemma inequality becomes as: $12r \le 4(R+r) \le 6R$, which follows from $R \ge 2r\ (Euler)$. Equality holds if and only if triangle is equilateral.

3) In ΔABC the following relationship holds:

$$36r \le \sum_{cyc}(b+c)\cot\frac{A}{2} \le \frac{9R^2}{r}$$

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Solution. Lemma. 4) In ΔABC the following relationship holds:

$$\sum_{cyc}(b+c)\cot\frac{A}{2} = \frac{2(s^2-r^2-4Rr)}{r}$$

Proof. Using identity $\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$ we get:

$$\sum_{cyc}(b+c)\cot\frac{A}{2}=\sum_{cyc}(b+c)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\sum_{cyc}(b+c)\frac{\sqrt{s(s-a)(s-b)(s-c)}}{(s-b)(s-c)}=$$

$$=F\cdot\sum_{cyc}\frac{b+c}{(s-b)(s-c)}=sr\cdot\frac{2(s^2-r^2-4Rr)}{sr^2}$$

$$=\frac{2(s^2-r^2-4Rr)}{r}, \text{which follows from}$$

$$\sum_{cyc}\frac{b+c}{(s-b)(s-c)}=\frac{2(s^2-r^2-4Rr)}{sr^2}$$

Let's get back to the main problem. Using **Lemma**, inequality can be written as:

$$36r\le\frac{2(s^2-r^2-4Rr)}{r}\le\frac{9R^2}{r}, \text{which is true from}$$

$$16Rr-5r^2\le s^2\le 4R^2+4Rr+3r^2(Gerretsen) \text{ and } R\ge 2r\ (Euler).$$

Equality holds if and only if triangle is equilateral.

5) In ΔABC the following relationship holds:

$$3\sum_{cyc}(b+c)\tan\frac{A}{2}\le\sum_{cyc}(b+c)\cot\frac{A}{2}$$

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Solution. Using these up **Lemmas**, we have that:

$$\sum_{cyc}(b+c)\tan\frac{A}{2}=4(R+r) \text{ and } \sum_{cyc}(b+c)\cot\frac{A}{2}=\frac{2(s^2-r^2-4Rr)}{r}$$

inequality becomes as: $3\cdot 4(R+r)\le\frac{2(s^2-r^2-4Rr)}{r}\Leftrightarrow s^2\ge 10Rr+7r^2$, which follows from $s^2\ge 16Rr-5r^2\ (Gerretsen)$ and $R\ge 2r\ (Euler)$.

Equality holds if and only if triangle is equilateral.

Reference:

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