THE HEPTAGONAL TRIANGLE REVISITED

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Abstract. In this paper are proved the characteristic metric relationships in the heptagonal triangle.

We call heptagonal triangle the obtuse scalene triangle whose vertices coincide with the first, second and fourth vertices of a regular heptagon (from an arbitrary starting vertex).
Thus its sides coincide with one side and the adjacent shorter and longer diagonals of the regular heptagon.
Lemma 1.
If in $\Delta ABC; a < b < c; \mu(B) = 2\mu(A); \mu(C) = 2\mu(B)$ then: $b^2 = a^2 + ac$;
$c^2 = b^2 + ba$.

Proof.

$$\mu(B) = 2\mu(A) \Rightarrow \sin B = \sin 2A$$
$$\sin B = 2\sin A\cos A \Rightarrow \frac{b}{2R} = 2\cdot\frac{a}{2R}\cdot\frac{b^2+c^2-a^2}{2bc}$$
$$b^2c = ab^2 + ac^2 - a^3 \Rightarrow b^2(c-a) - s(c^2-a^2) = 0$$
$$b^2(c-a) - a(c-a)(c+a) = 0$$
$$(c-a)(b^2 - a(c+a)) = 0; c - a \neq 0$$
$$b^2 = a^2 + ac$$

Analgous:

$$\mu(C) = 2\mu(B) \Rightarrow c^2 = b^2 + ba$$

□

Lemma 2.
If in $\Delta ABC; a < b < c; \mu(C) = 2\mu(A) + \mu(B)$ then: $a^2 = c^2 - bc$.

Proof.

$$\mu(C) = 2\mu(A) + \mu(B) \Rightarrow 2\mu(C) = 2\mu(A) + \mu(B) + \mu(C)$$
$$2\mu(C) = \pi + \mu(A)$$
$$\sin 2C = \sin(\pi + A)$$
$$2\sin C\cos C = -\sin A$$
$$2\cdot\frac{c}{2R}\cdot\frac{a^2+b^2-c^2}{2ab} = -\frac{a}{2R}$$
$$ca^2 + cb^2 - c^3 = -a^2b$$
$$a^2(b+c) - c(c^2-b^2) = 0$$
$$a^2(b+c) - c(b+c)(c-b) = 0$$
$$a^2 - c^2 + bc = 0$$
$$a^2 = c^2 - bc$$

If $\mu(A) = \frac{\pi}{7}; \mu(B) = \frac{2\pi}{7}; \mu(C) = \frac{4\pi}{7}$ by lemma 1 and lemma 2 we obtain that for the triangle with sides $a, b, c; a < b < c$ the relationships:

$$b^2 = a^2 + ac; c^2 = b^2 + ba; a^2 = c^2 - bc \tag{1}$$

The heptagonal triangle with sides $a, b, c; a < b < c$ and angles A, B, C verify (1). By adding relationships (1) we obtain:

$$b^2 + c^2 + a^2 = a^2 + ac + b^2 + ba + c^2 - bc$$

$$bc = ba + ac$$

$$\frac{1}{a} = \frac{b+c}{bc}$$

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} \tag{2}$$

By multiplying (2) with $2F$:

$$\frac{2F}{a} = \frac{2F}{b} + \frac{2F}{c} \Rightarrow h_a = h_b + h_c$$

□

Lemma 3.

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{8} \tag{3}$$

Proof. We will use the well known identity:

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}; n \in \mathbb{N}; n \geq 2$$

For $n = 7$:

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\frac{4\pi}{7}\sin\frac{5\pi}{7}\sin\frac{6\pi}{7} = \frac{7}{2^6}$$

$$\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}\sin\left(\pi - \frac{4\pi}{7}\right)\sin\left(\pi - \frac{5\pi}{7}\right)\sin\left(\pi - \frac{6\pi}{7}\right) = \frac{7}{64}$$

$$\left(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\right)^2 = \frac{7}{64} \Rightarrow \sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} = \frac{\sqrt{7}}{8}$$

□

Lemma 4.
The equation:

$$64y^3 - 112y^2 + 56y - 7 = 0 \tag{4}$$

has the roots:

$$x_1 = \sin^2\frac{\pi}{7}; x_2 = \sin^2\frac{2\pi}{7}; x_3 = \sin^2\frac{4\pi}{7}$$

Proof. We will prove that $x_1 = \sin^2\frac{\pi}{7}$ is a solution for the equation (4):

$$\sin\frac{4\pi}{7} = \sin\left(\pi - \frac{4\pi}{7}\right) = \sin\frac{3\pi}{7}$$

$$\sin^2\frac{4\pi}{7} = \sin^2\frac{3\pi}{7}$$

$$\left(2\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\right)^2 = \left(\sin\frac{\pi}{7}\left(3 - 4\sin^2\frac{\pi}{7}\right)\right)^2$$

$$4\sin^2\frac{2\pi}{7}\cos^2\frac{2\pi}{7}=\sin^2\frac{\pi}{7}\Big(3-4\sin^2\frac{\pi}{7}\Big)^2$$
$$4\cdot 4\sin^2\frac{\pi}{7}\cos^2\frac{\pi}{7}\Big(1-2\sin^2\frac{\pi}{7}\Big)^2=\sin^2\frac{\pi}{7}\Big(3-4\sin^2\frac{\pi}{7}\Big)^2$$

Denote:

$$\sin^2\frac{\pi}{7}=y$$
$$16\Big(1-\sin^2\frac{\pi}{7}\Big)\Big(1-2\sin^2\frac{\pi}{7}\Big)^2=\Big(3-4\sin^2\frac{2\pi}{7}\Big)^2$$
$$16(1-y)(1-2y)^2=(3-4y)^2$$
$$16(1-y)(1-4y+4y^2)=(3-4y)^2$$
$$16(1-4y+4y^2-y+4y^2-4y^3)=(3-4y)^2$$
$$16-80y+128y^2-64y^3=9+16y^2-24y$$
$$7-56y+112y^2-64y^3=0$$
$$64y^3-112y^2+56y-7=0$$
$$S_1=x_1+x_2+x_3=-\frac{-112}{64}=\frac{14}{8}=\frac{7}{4}$$

(5)
$$\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}=\frac{7}{4}$$
$$S_2=x_1x_2+x_2x_3+x_3x_1=\frac{56}{64}=\frac{14}{16}=\frac{7}{8}$$

(6)
$$\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}+\sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7}+\sin^2\frac{4\pi}{7}\sin^2\frac{\pi}{7}=\frac{7}{8}$$

(7)
$$S_3=x_1x_2x_3=\Big(\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\Big)^2=\frac{7}{64}$$

R - circumradii, F - area;

$$a=2R\sin A=2R\sin\frac{\pi}{7}$$
$$b=2R\sin B=2R\sin\frac{2\pi}{7}$$
$$c=2R\sin C=2R\sin\frac{4\pi}{7}$$
$$F=\frac{abc}{4R}=\frac{8R^3\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}}{4R}\overset{(4)}{=}2R^2\cdot\frac{\sqrt{7}}{8}=\frac{\sqrt{7}}{4}R^2$$
$$a^2+b^2+c^2=4R^2\Big(\sin^2\frac{\pi}{7}+\sin^2\frac{2\pi}{7}+\sin^2\frac{4\pi}{7}\Big)\overset{(5)}{=}4R^2\cdot\frac{7}{4}=7R^2$$

(8)
$$a^2+b^2+c^2=7R^2$$
$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{4R^2}\Big(\frac{1}{\sin^2\frac{\pi}{7}}+\frac{1}{\sin^2\frac{2\pi}{7}}+\frac{1}{\sin^2\frac{4\pi}{7}}\Big)=$$
$$=\frac{1}{4R^2}\cdot\frac{\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}+\sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7}+\sin^2\frac{4\pi}{7}\sin^2\frac{\pi}{7}}{\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{4\pi}{7}}=$$
$$\overset{(6);(7)}{=}\frac{1}{4R^2}\cdot\frac{\frac{7}{8}}{\frac{7}{64}}=\frac{1}{4R^2}\cdot 8=\frac{2}{R^2}$$

(9)
$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{2}{R^2}$$

$$h_a^2 + h_b^2 + h_c^2 = 4F^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) =$$

$$= 4 \cdot \frac{7}{16}R^4 \cdot \frac{2}{R^2} = \frac{7}{2}R^2 = \frac{7R^2}{2} \overset{(8)}{=} \frac{a^2+b^2+c^2}{2}$$

$$h_a^2 + h_b^2 + h_c^2 = \frac{a^2+b^2+c^2}{2}$$

In conclusion, in a heptagonal triangle with sides $a < b < c; \mu(A) = \frac{\pi}{7}$; $\mu(B) = \frac{2\pi}{7}; \mu(C) = \frac{4\pi}{7}$ the following relationship holds:

$$a^2 = c^2 - bc; b^2 = a^2 + ac; c^2 = b^2 + ba$$

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c};$$

$$h_a = h_b + h_c; h_a^2 + h_b^2 + h_c^2 = \frac{a^2+b^2+c^2}{2}$$

$$F = \frac{\sqrt{7}}{4}R^2;$$

$$a^2 + b^2 + c^2 = 7R^2; \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}$$

□

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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