SPECIAL LIMITS WITH RIEMANN'S SUMS

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Abstract. In this paper it is developed a method for calculus of sequences' limits using Riemann's sums.

Main result:

If $\alpha, \beta \in \mathbb{R}; \beta \neq 0; \alpha < \beta; f : [\alpha, \alpha + \beta] \to \mathbb{R}, f$ continuous then:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} f\left(\alpha + \frac{i\beta}{n}\right) f\left(\alpha + \frac{j\beta}{n}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)dx\right)^2 \tag{1}$$

Proof. Let be:

$$\Delta_n = \left(\alpha < \alpha + \frac{\beta}{n} < \alpha + \frac{2\beta}{n} < \ldots < \alpha + \frac{(n-1)\beta}{n} < \beta\right)$$

Denote $x_n^i = \xi_n^i = \alpha + \frac{i\beta}{n}$ □

$$||\Delta_n|| = x_n^i - x_n^{i-1} = \alpha + \frac{i\beta}{n} - \alpha - \frac{(i-1)\beta}{n} = \frac{\beta}{n}$$

$$\lim_{n\to\infty} ||\Delta_n|| = \lim_{n\to\infty} \frac{\beta}{n} = 0$$

$$x_n^{i-1} \le \xi_n^i \le x_n^i$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\alpha + \frac{i\beta}{n}\right) = \lim_{n\to\infty} \frac{1}{\beta} \cdot \frac{\beta}{n} \sum_{i=1}^{n} f\left(\alpha + \frac{i\beta}{n}\right) =$$

$$= \frac{1}{\beta} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\alpha + \frac{i\beta}{n}\right)\left(\alpha + \frac{i\beta}{n} - \alpha - \frac{(i-1)\beta}{n}\right) =$$

$$= \frac{1}{\beta} \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_n^i)(x_n^i - x_n^{i-1}) =$$

$$= \frac{1}{\beta} \lim_{n\to\infty} \sigma_{\Delta_n}(f_1 \xi^n) = \frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)dx \tag{2}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} f^2\left(\alpha + \frac{i\beta}{n}\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n} \sum_{i=1}^{n} f^2\left(\alpha + \frac{i\beta}{n}\right)\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f^2\Big(\alpha + \frac{i\beta}{n}\Big) =$$

$$= 0 \cdot \left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)dx\right) = 0$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} f^2\Big(\alpha + \frac{i\beta}{n}\Big) = 0 \tag{3}$$

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} f\Big(\alpha + \frac{i\beta}{n}\Big) f\Big(\alpha + \frac{j\beta}{n}\Big) =$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[\left(\frac{1}{n}\sum_{i=1}^{n} f\Big(\alpha + \frac{i\beta}{n}\Big)\right)^2 - \frac{1}{n^2}\sum_{i=1}^{n} f^2\Big(\alpha + \frac{i\beta}{n}\Big)\right] =$$

$$\overset{(2);(3)}{=} \frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)dx\right)^2 - 0 = \frac{1}{2}\left(\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} f(x)dx\right)^2$$

Corollary 1:
If $\alpha, \beta \in \mathbb{R}; \beta \neq 0; \alpha < \beta$ then:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \Big(\alpha + \frac{i\beta}{n}\Big)\Big(\alpha + \frac{j\beta}{n}\Big) = \frac{\beta^2 + 2\alpha\beta}{8\beta^2}$$

Proof. We take in (1) : $f(x) = x$. □

Corollary 2:
If $\alpha, \beta \in \mathbb{R}; \beta \neq 0; \alpha < \beta$ then:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \cos\Big(\alpha + \frac{i\beta}{n}\Big) \cos\Big(\alpha + \frac{j\beta}{n}\Big) =$$

$$= \frac{(\sin(\alpha+\beta) - \sin\alpha)^2}{2\beta^2} \tag{2}$$

Proof. We take in (1) : $f(x) = \cos x$. □

Corollary 3:
If $0 < \alpha < \beta$ then:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \frac{1}{(\alpha + \frac{i\beta}{n})(\alpha + \frac{j\beta}{n})} = \frac{1}{2\alpha^2(\alpha+\beta)^2}$$

Proof. We take in (1) : $f(x) = \frac{1}{x}$. □

Corollary 4:

$$\lim_{n\to\infty} \frac{1}{n^4} \sum_{1\le i<j\le n} ij = \frac{1}{8}$$

Proof. We take in (1) : $f(x) = x, \alpha = 0, \beta = 1$. □

Corollary 5:

$$\lim_{n\to\infty} \frac{1}{n^6} \sum_{1\le i<j\le n} i^2 j^2 = \frac{1}{18}$$

Proof. We take in $(1) : f(x) = x^2; \alpha = 0; \beta = 1.$ □

Corrolary 6:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \cos i \cdot \cos j = \frac{\sin^2 1}{2}$$

We take in $(2) : \alpha = 0; \beta = 1.$
Proposed problems:
1. If $0 < \alpha < \beta$ then find:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \ln\Big(\alpha + \frac{i\beta}{n}\Big) \ln\Big(\alpha + \frac{j\beta}{n}\Big)$$

2. If $\alpha, \beta \in \mathbb{R}; \beta \neq 0; \alpha < \beta$ then find:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \sin\Big(\alpha + \frac{i\beta}{n}\Big) \sin\Big(\alpha + \frac{j\beta}{n}\Big)$$

3. If $0 < \alpha < \beta$ then find:

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \frac{1}{(\alpha + \frac{i\beta}{n})^3(\alpha + \frac{j\beta}{n})^3}$$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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