RMM - Abstract Algebra Marathon 201 - 300

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201. **If $A, B \in M_3(\mathbb{R})$, $det((AB-BA)^2+AB-BA+I_3)=0$ then find:**

$$\Omega = det(AB-BA)$$

Proposed by Marian Ursărescu-Romania

Solution by Ravi Prakash-New Delhi-India

Let $C = AB - BA$, $tr(C) = 0$.

We are given $det(C^2+C+I_3)=0$ hence,

$$det\left[(C-\omega I_3)\overline{(C+\omega I_3)}\right]=0$$

$$|det(C-\omega I_3)|^2=0 \Rightarrow det(C-\omega I_3)=0;\ \ (1)$$

Let $det(C-tI_3) = -t^3+tr(C)t^2-\alpha t+det(C) = -t^3-\alpha t+det(C)$

$\therefore (1)$ gives: $-\omega^3-\alpha\omega+det(C)=0 \Rightarrow det(C)-1+\frac{\alpha}{2}=0$ and

$$\frac{\sqrt{3}}{2}\alpha = 0 \Rightarrow \alpha = 0 \Rightarrow det(C)=1$$

202. **Let $x^{2020}+a_{2019}x^{2019}+a_{2018}x^{2018}+\cdots+a_0 \in \mathbb{Z}[x]$ and all roots of this polynomials are positive real numbers.**

Find the smallest possible value of coefficient a_{1010}.

Proposed by Gantumur Choijilsuren-Mongolia

Solution by Abdul Hannan-Tezpur-India

Let $x^{4n}+a_{4n-1}x^{4n-1}+\cdots+a_1x+a_0 \in \mathbb{Z}[x]$ and all roots of this polynomial are positive real numbers. Find the smallest possible value of coefficient a_{2n}.

Here is a small observation that we will need later:

$$\frac{\binom{4n-1}{2n-1}}{\binom{4n}{2n}} = \frac{(4n-1)!}{(2n)!\,(2n-1)!}\cdot\frac{(2n)!\,(2n)!}{(4n)!} = \frac{2n}{4n} = \frac{1}{2}$$

Let $\beta_1, \beta_2, \beta_3, \dots, \beta_{4n-1}, \beta_{4n}$ be the (positive real) roots. Then $a_0 = \beta_1\beta_2\cdot\ldots\cdot\beta_{4n} > 0$.

Being an integer, we must have, $a_0 \ge 1$. Also, we have

$$a_{2n} = \sum_{1\le i_1\le i_2\le\cdots\le i_{2n}\le 4n} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_{2n}} \overset{AGM}{\ge}$$

$$\overset{AGM}{\geq}\binom{4n}{2n}\left(\prod_{1\le i_1\le i_2\le\cdots\le i_{2n}\le 4n}\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{2n}}\right)^{\frac{1}{\binom{4n}{2n}}}=$$

$$=\binom{4n}{2n}\left(\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{2n}}\right)^{\frac{\binom{4n-1}{2n-1}}{\binom{4n}{2n}}}=\binom{4n}{2n}\sqrt{\beta_{i_1}\beta_{i_2}\cdots\beta_{i_{2n}}}=\binom{4n}{2n}\sqrt{a_0}\ge\binom{4n}{2n};a_0\ge 1$$

If we put $\beta_1=\beta_2=\cdots=\beta_{4n}=1$, then we see that

$$a_{2n}=\sum_{1\le i_1\le i_2\le\cdots\le i_{2n}\le 4n}1=\binom{4n}{2n}$$

Since $\binom{4n}{2n}$ is achieved, it is indeed the minimum value of a_{2n}.

203. If $A\in M_3(\mathbb{R}),det(A^2-3A+3I_3)=0$ then:

$$2det(A^2+3I_3)\ge 3\left(3\sqrt{3}+detA\right)^2$$

Proposed by Marian Ursărescu-Romania

Solution 1 by George Florin Şerban-Romania

$$P(\lambda)=\lambda^3-(trA)\lambda^2+(trA^*)\lambda-detA=-det(A-\lambda I_3)$$

$$x^2-3x+2=0,x_{1,2}=\frac{3\pm i\sqrt{3}}{2},x_1^2=\frac{3+3i\sqrt{3}}{2},x_1^3=3i\sqrt{3}$$

$$det(A^2-3A+3I_3)=det(A-x_1I_3)det(A-x_2I_3)=P(x_1)P(x_2)=0$$

$$\Rightarrow P(x_1)=0 \text{ or } P(x_2)=0$$

$$P(x_1)=x_1^3-(trA)x_1^2+(trA^*)x_1-detA$$

$$=3i\sqrt{3}-\frac{3+3i\sqrt{3}}{2}(trA)+\frac{3+3i\sqrt{3}}{2}(trA^*)-detA=0\Rightarrow$$

$$\begin{cases}-3trA+3trA^*-2detA=0\\6\sqrt{3}-3\sqrt{3}trA+\sqrt{3}trA^*=0\end{cases}\Rightarrow\begin{cases}-3trA+3trA^*-2detA=0\\trA^*=3trA-6\end{cases}\Rightarrow$$

$$detA=3trA-9$$

$$2det(A^2+3I_3)=2det\left(A+i\sqrt{3}I_3\right)det\left(A-i\sqrt{3}I_3\right)=$$

$$=2P\left(i\sqrt{3}\right)P\left(-i\sqrt{3}\right)=2\left[(3trA-detA)^2-\left(3\sqrt{3}i-trA^*i\sqrt{3}\right)^2\right]=$$

$$= 2 \cdot 3(det^2A + 27) \geq 9\left(detA + 3\sqrt{3}\right)^2$$

$$\Rightarrow 2det(A^2 + 3I_3) \geq 3\left(3\sqrt{3} + detA\right)^2$$

Solution 2 by Ravi Prakash-New Delhi-India

$$det(A^2 - 3A + 3I_3) = 0 \Rightarrow det\big((\alpha I_3 - A)(\overline{\alpha}I_3 - A)\big) = 0, \alpha = \frac{1}{2}(3 + \sqrt{3}i)$$

$$\Rightarrow det(\alpha I_3 - A)det(\overline{\alpha I_3 - A}) = 0 \Rightarrow |det(\alpha I_3 - A)|^2 = 0 \Rightarrow det(\alpha I_3 - A) = 0$$

Similarly, $det(\overline{\alpha}I_3 - A) = 0$, $\alpha, \overline{\alpha}$ –eigen values of A.

Let λ be the third eigen value of A. Characteristic equation of A is:

$$det(tI_3 - A) = P(t) = t^3 - (\alpha + \overline{\alpha} + \lambda)t^2 + (\lambda(\alpha + \overline{\alpha}) + \alpha\overline{\alpha})t - \alpha\overline{\alpha}\lambda = 0$$

$P(t) = t^3 - (3 + \lambda)t^2 + (3\lambda + 3)t - 3\lambda = 0$. Now,

$det(A^2 + 3I_3) = \left|det(\sqrt{3}iI_3 - A)\right|^2 = \left|P(\sqrt{3}i)\right|^2$. We have:

$$P(\sqrt{3}i) = 9 + 3\sqrt{3}\lambda i \Rightarrow \left|P(\sqrt{3}i)\right|^2 = 3(27 + (detA)^2); (detA = 3\lambda)$$

$$2det(A^2 + 3I_3) = 6(27 + (detA)^2) = 3\left(\left(3\sqrt{3} + detA\right)^2 + \left(3\sqrt{3} - detA\right)^2\right)$$

$$\geq 3\left(3\sqrt{3} + detA\right)^2$$

204. If $x, y, z \in \mathbb{R} - \left\{(2k + 1)\frac{\pi}{2} \mid k \in \mathbb{Z}\right\}$ then prove:

$$2\prod_{cyc} cosx \cdot \sum_{cyc} sinxsin(y - z)tanx + \sum_{cyc} sinxsin(y - z)sin(y + z - x) = 0$$

Proposed by Florică Anastase-Romania

Solution by Adrian Popa-Romania

$$\sum_{cyc} sinxsin(y - z)tanx = \sum_{cyc} \frac{sin(y - z)}{cosx} - \sum_{cyc} cosxsin(y - z)$$

But

$$\sum_{cyc} cosxsin(y - z) = 0$$

Hence,

$$\sum_{cyc}\frac{sin(y-z)}{cosx}=\frac{1}{\prod cosx}\cdot\sum_{cyc}cosycoszsin(y-z)=$$

$$=\frac{1}{2\prod cosx}\cdot\left[\sum_{cyc}cos(y+z)sin(y-z)+\frac{1}{2}\sum_{cyc}sin2(y-z)\right];\ (1)$$

Now,

$$\frac{1}{2}\sum_{cyc}sin2(y-z)=-2sin(x-y)sin(y-z)sin(z-x)$$

Then,

$$\sum_{cyc}sinxsin(y-z)tanx+\frac{sin(x-y)sin(y-z)(sin(z-x)}{\prod cosx}=0;\ (2)$$

On the other hand, we have:

$$sinxsin(y-z)sin(y+z-x)=sin(y-z)\cdot\frac{2sinxsin(y+z-x)}{2}=$$

$$=sin(y-z)\cdot\frac{cos(y+z-2x)-cos(y+z)}{2}=$$

$$=\frac{1}{2}sin(y-z)cos(y+z-2x)-\frac{1}{2}sin(y-z)cos(y+z)=$$

$$=\frac{1}{4}[sin2(y-x)+sin2(x-z)]-\frac{1}{4}(sin2y-sin2z)$$

Hence,

$$\sum_{cyc}sinxsin(y-z)sin(y+z-x)=-\frac{1}{2}\sum_{cyc}sin2(y-z)=$$

$$=-\frac{sin2(y-z)+sin2(z-x)+sin2(x-y)}{2}=$$

$$=-sin(y-x)cos(x+y-2z)-sin(x-y)cos(x-y)=$$

$$=sin(x-y)[cos(x+y-2z)-cos(x-y)]=$$

$$=2sin(x-y)sin(y-z)sin(z-x);\ (3)$$

From $(1),(2),(3)$ ***we get:***

$$2\prod_{cyc}cosx\cdot\sum_{cyc}sinxsin(y-z)tanx+\sum_{cyc}sinxsin(y-z)sin(y+z-x)=0$$

205. Prove that the number:

$$\frac{\sqrt[4]{3\sqrt[3]{2}+3}-\sqrt[4]{\sqrt[3]{2}-1}}{\sqrt[4]{3\sqrt[3]{2}+3}-2\sqrt[4]{\sqrt[3]{2}-1}}$$

is a solution of the equation: $x^3-6x^2+6x-2=0$.

Proposed by Vasile Mircea Popa-Romania

Solution by Abdul Hannan-Tezpur-India

$$\text{Let: } x=\frac{\sqrt[4]{3\sqrt[3]{2}+3}-\sqrt[4]{\sqrt[3]{2}-1}}{\sqrt[4]{3\sqrt[3]{2}+3}-2\sqrt[4]{\sqrt[3]{2}-1}} \text{ and } u=\sqrt[4]{\frac{3+3\sqrt[3]{2}}{\sqrt[3]{2}-1}}$$

$$\text{Let } a=\sqrt[3]{2}. \text{ Then } u^4=\frac{3(a+1)}{a-1}=\frac{3(a+1)^4}{(a+1)^3(a-1)}=\frac{3(a+1)^4}{(a^3+3a^2+3a+1)(a-1)}=$$

$$\overset{a^3=2}{=}\frac{3(a+1)^4}{(3a^2+3a+3)(a-1)}=\frac{(a+1)^4}{(a^2+a+1)(a-1)}=\frac{(a+1)^4}{a^3-1}$$

$$\overset{a^3=2}{=}(a+1)^4$$

Since $u>0$, we must have: $u=a+1=\sqrt[3]{2}+1$

$$\text{Now, } x=\frac{u-1}{u-2}\Rightarrow\frac{2x-1}{x-1}=u=\sqrt[3]{2}+1\Rightarrow\frac{x}{x-1}=\sqrt[3]{2}$$

$$\Rightarrow 2(x-1)^3=x^3\Rightarrow x^3-6x^2+6x-2=0$$

206. $A(a), B(b), C(c), a,b,c\in\mathbb{C}^*$ –different pairs, $|a|=|b|=|c|=1$. Prove that:

$$\sum_{cyc}|2a+b+c|^2=3\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution by Ravi Prakash-New Delhi-India

Let $z=a+b+c$

$$|2a+b+c|^2=|a+z|^2=|a|^2+\bar{a}z+a\bar{z}+|z|^2$$

$$|a+2b+c|^2=|b+z|^2=|b|^2+\bar{b}z+b\bar{z}+|z|^2$$

$$|a+b+2c|^2=|c+z|^2=|c|^2+\bar{c}z+c\bar{z}+|z|^2$$

Adding, we get:

$$3=\sum_{cyc}|2a+b+c|^2=3+(\overline{a}+\overline{b}+\overline{c})z+(a+b+c)\overline{z}+3|z|^2$$

$$0=\overline{z}z+z\overline{z}+3|z|^2=5|z|^2\Rightarrow z=0.$$

Now,

$$|b-a|^2+|b=a|^2=2|a|^2+2|b|^2=0\Rightarrow |b-a|^2+|-c|^2=4$$

$\Rightarrow |b-a|^2=3$ or $AB=\sqrt{3}$. Similarly: $CA=BC=\sqrt{3}$. Therefore, $AB=BC=CA$.

207. $A(a),B(b),C(c),a,b,c\in\mathbb{C}^*$ −**different in pairs,** $|a|=|b|=|c|=1$.

Prove that:

$$\sum_{cyc}\left(\left|a^2+ab+bc+ca\right|+|a-b|^2\right)=12\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution by Iulian Cristi-Romania

Applying the real product of complex number $R=1\Rightarrow$

$$a^2=a\cdot a=|a|^2=1;ab=1-\frac{1}{2}|a-b|^2;bc=1-\frac{1}{2}|b-c|^2;ca=1-\frac{1}{2}|c-a|^2$$

Let $\alpha=|a-b|,\beta=|b-c|,\gamma=|c-a|$.

Having the circumcenter O of the triangle as the origin of the complex plane, hence

$$6\left|4-\frac{\alpha^2+\beta^2+\gamma^2}{2}\right|+\alpha^2+\beta^2+\gamma^2=12$$

We know that $\alpha^2+\beta^2+\gamma^2=9(1-OG^2)$

$$6\left|4-\frac{9}{2}(1-OG^2)\right|+9(1-OG^2)=12,$$

$$6\left(\frac{9}{2}(1-OG^2)-4\right)+9(1-OG^2)=12$$

$$36=9(1-OG^2)+27(1-OG^2)=12$$

$$(1-OG^2)36=36\Leftrightarrow G=O\Leftrightarrow AB=BC=CA.$$

208. $a,b,c\in\mathbb{C}^*$ −**different pairs,** $|a|=|b|=|c|=1,A(a),B(b),C(c)$.

Prove that:

$$\sum_{cyc}|a+b-2c|=\sum_{cyc}\left|a^2-ab-ac+bc\right|\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution by Florentin Vişescu-Romania

$$\sum_{cyc}|a+b-2c| = \sum_{cyc}|a+b+c-3c| = \sum_{cyc}|a(a-b)-c(a-c)|$$

$$\Leftrightarrow 3\sum_{cyc}\left|\frac{a+b+c}{3}-c\right| = \sum_{cyc}|a-b|\cdot|a-c| \Leftrightarrow 3\sum_{cyc}AG = \sum_{cyc}AB\cdot AC$$

$$\Leftrightarrow 3\sum_{cyc}\frac{2}{3}m_a = \sum_{cyc}bc \Leftrightarrow 2(m_a+m_b+m_c) = ab+bc+ca$$

We know that: $m_a \ge \frac{b^2+c^2}{4R} \Rightarrow m_a+m_b+m_c \ge \frac{a^2+b^2+c^2}{2R}$

$$\Leftrightarrow 2(m_a+m_b+m_c) \ge \frac{a^2+b^2+c^2}{R}$$

But $R=1 \Rightarrow 2(m_a+m_b+m_c) \ge a^2+b^2+c^2$

$$\Rightarrow ab+bc+ca \ge a^2+b^2+c^2$$

$$\Leftrightarrow 2a^2+2b^2+2c^2-2ab-2bc-2ca \le 0$$

$$\Leftrightarrow (a-b)^2+(b-c)^2+(c-a)^2 \le 0 \Leftrightarrow a=b=c \Rightarrow AB=BC=CA.$$

209. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs, such that $|z_1| = |z_2| = |z_3|$. If

$$\sum\frac{|z_2-z_3|}{|z_2+z_3-2z_1|} = \sqrt{3}$$

then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu-Romania

Solution by proposer

Let be $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(O,R)$

$$\sum\frac{|z_2-z_3|}{2\left|z_1-\frac{z_2+z_3}{2}\right|} = \sqrt{3} \Leftrightarrow \sum\frac{|z_2-z_3|}{\left|z_1-\frac{z_2+z_3}{2}\right|} = 2\sqrt{3} \Leftrightarrow$$

$$\sum\frac{a}{m_a} = 2\sqrt{3};\ \ (1)$$

But $\sum\frac{a}{m_a} \ge 2\sqrt{3};\ \ (2)$

From $(1)\&(2) \Rightarrow \Delta ABC$ $-$equilqteral.

$$(2) \Leftrightarrow m_a \le \frac{a^2+b^2+c^2}{2\sqrt{3}a}$$

210. Let be $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs, such that $|z_1| = |z_2| = |z_3|$. If

$$\sum \left|\frac{(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|}{z_2+z_3-2z_1}\right|^2 = \frac{(|z_1-z_2|+|z_2-z_3|+|z_3-z_1|)^2}{3}$$

then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu-Romania

Solution by proposer

Let be $A(z_1), B(z_2), C(z_3) \Rightarrow \Delta ABC \subset C(O,R), |z_1| = |z_2| = |z_3| = R$

$|z_1-z_2| = AB = c, |z_2-z_3| = BC = a, |z_3-z_1| = AC = b$

$$\sum \left|\frac{\frac{b(z_1-z_3)+c(z_1-z_3)}{a+b+c}}{\frac{z_2+z_3-2z_1}{3}}\right|^2 = 3 \Leftrightarrow \sum \left|\frac{\frac{(b+c)z_1-bz_2-cz_3}{a+b+c}}{\frac{z_1+z_2+z_3-3z_1}{3}}\right|^2 = 3$$

$$\Leftrightarrow \sum \left|\frac{\frac{(a+b+c)z_1-az_1-bz_2-cz_3}{a+b+c}}{\frac{z_1+z_{2+z_3}}{3}-z_1}\right|^2 = 3 \Leftrightarrow \sum \left|\frac{z_1-\frac{az_1+bz_2+cz_3}{a+b+c}}{\frac{z_1+z_2+z_3}{3}-z_1}\right|^2 = 3$$

$$\Leftrightarrow \sum \frac{AI^2}{GA^2} = 3; \quad (1)$$

But in any ΔABC: $\sum \frac{AI^2}{GA^2} \le 3; (2)$

Equality holds if and only if triangle is equilateral.

From $(1)\&(2) \Rightarrow \Delta ABC$ –equilateral.

$$(2) \Leftrightarrow \sum \frac{AI^2}{\frac{4}{9}m_a^2} \le 3 \Leftrightarrow \sum \frac{AI^2}{m_a^2} \le \frac{4}{3}, \text{but } m_a \ge \sqrt{s(s-a)} \Rightarrow$$

$$\sum \frac{AI^2}{s(s-a)} \le \frac{4}{3} \Leftrightarrow \frac{s^2+r^2+4Rr}{s^2} \le \frac{4}{3} \Leftrightarrow s^2 \ge 12Rr + 3r^2$$

which is true from $s^2 \ge 16Rr - 5r^2 (Gerretsen)$

211. **Let $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs, such that $|z_1| = |z_2| = |z_3|$, If**

$$\sum |(2z_1 - z_2 - z_3)(2z_2 - z_1 - z_3)(z_1 - z_2)| = 9|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|$$

Then z_1, z_2, z_3 are the affixes of an equilateral triangle.

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

Let be $A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(O,R), |z_1| = |z_2| = |z_3| = R$

$$\sum 4\left|z_1 - \frac{z_2 + z_3}{2}\right| \cdot \left|z_2 - \frac{z_1 + z_3}{2}\right| \cdot |z_1 - z_2| = 9|(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)|$$

$$\Leftrightarrow \sum m_a m_b \cdot c = \frac{9}{4} abc \Leftrightarrow \sum \frac{m_a m_b}{ab} = \frac{9}{4}; (1)$$

But in any ΔABC: $\frac{m_a m_b}{ab} + \frac{m_b m_c}{bc} + \frac{m_c m_a}{ac} \ge \frac{9}{4}; (2)$

From $(1)\&(2)$ equality holds if and only if triangle is equilateral.

$\frac{m_a m_b}{ab} + \frac{m_b m_c}{bc} + \frac{m_c m_a}{ac} \ge \frac{9}{4}; (2)$ follows from

$$a \cdot PB \cdot PC + b \cdot PC \cdot PA + c \cdot PA \cdot PB \ge abc$$

Equality holds if and only if triangle is equilateral.

Solution 2 by Florentin Vişescu-Romania

$$\sum_{cyc} |3z_1 - (z_1 + z_2 + z_3)| \cdot |3z_2 - (z_1 + z_2 + z_3)| \cdot |z_1 - z_2| = 9\prod_{cyc} |z_1 - z_2|$$

$$9\sum_{cyc} \left|z_1 - \frac{(z_1 + z_2 + z_3)}{3}\right| \cdot \left|z_2 - \frac{(z_1 + z_2 + z_3)}{3}\right| \cdot |z_1 - z_2| = 9\prod_{cyc} |z_1 - z_2|$$

$$9\sum_{cyc} AG \cdot BG \cdot AB = 9\prod_{cyc} AB \Leftrightarrow$$

$$9 \cdot \frac{4}{9}\sum_{cyc} m_a m_b \cdot c = 9\prod_{cyc} c \Leftrightarrow \sum_{cyc} m_a m_b \cdot c = \prod_{cyc} c \Leftrightarrow$$

$$\sum_{cyc} \frac{m_a m_b}{ab} = \frac{9}{4}$$

But in any ΔABC: $\sum_{cyc} \frac{m_a m_b}{ab} \ge \frac{9}{4}$;

Equality holds if and only if $a = b = c \Rightarrow AB = BC = CA$.

Solution 3 by George Florin Şerban-Romania

Lemma. Let $z_1, z_2, z_3 \in \mathbb{C}^*$ different in pairs, then:

$$\sum_{cyc} \frac{\left(z_2 - \frac{z_1 + z_3}{2}\right)\left(z_3 - \frac{z_1 + z_2}{2}\right)}{(z_1 - z_2)(z_1 - z_3)} = \frac{9}{4}$$

Proof.

$$\sum_{cyc} \frac{(2z_2 - (z_1 + z_3))(2z_3 - (z_1 + z_2))}{4(z_1 - z_2)(z_1 - z_3)} = \frac{9}{4} \Leftrightarrow$$

$$\sum_{cyc} (4z_2z_3 - 2z_1z_2 - 2z_2^2 - 2z_1z_3 + z_1^2 + z_1z_2 - z_1z_3 + z_2z_3)(z_3 - z_2)$$

$$= 9(z_1 - z_2)(z_2 - z_3)(z_3 - z_1) - \text{true}.$$

$$\frac{9}{4} = \sum_{cyc} \frac{\left(z_2 - \frac{z_1 + z_3}{2}\right)\left(z_3 - \frac{z_1 + z_2}{2}\right)}{(z_1 - z_2)(z_1 - z_3)} \le \sum_{cyc} \frac{\left|z_2 - \frac{z_1 + z_3}{2}\right| \cdot \left|z_3 - \frac{z_1 + z_2}{2}\right|}{|z_1 - z_2| \cdot |z_1 - z_3|} =$$

$$= \sum_{cyc} \frac{m_{b'} m_{c'}}{b'c'} \Rightarrow \sum_{cyc} \frac{m_{b'} m_{c'}}{b'c'} \ge \frac{9}{4}$$

where $A(z_1) = A(a), B(z_2) = B(b), C(z_3) = C(c)$

Equality holds if and only if $a' = b' = c'$

ΔABC equilateral, $AB = c', BC = b', AB = c'$

$M \in (BC), N \in (CA)$, –midpoints $\Rightarrow M\left(\frac{b+c}{2}\right), N\left(\frac{a+c}{2}\right)$ then

$$AM = \left|a - \frac{b + c}{2}\right| = \frac{|2a - b - c|}{2} = m_a \Rightarrow |2a - b - c| = 2m_a$$

$$BN = \left|b - \frac{a + c}{2}\right| = \frac{|2b - a - c|}{2} = m_b \Rightarrow |2b - a - c| = 2m_b$$

$$\sum_{cyc} |(2a - b - c)(2b - c - a)(a - b)| = 4 \sum_{cyc} m_{a'} m_{b'} \cdot c' =$$

$$= 4a'b'c' \sum_{cyc} \frac{m_{a'}}{a'} \cdot \frac{m_{b'}}{b'} = 9 \left| \prod_{cyc} (a - b) \right| = 9a'b'c' \Rightarrow$$

$$\sum_{cyc}\frac{m_{a'}}{a'}\cdot\frac{m_{b'}}{b'}=\frac{9}{4}\Rightarrow a'=b'=c'\Rightarrow AB=BC=CA$$

Note by editor: Bager's inequality-1971:

$$\sum_{cyc}\frac{m_a m_b}{ab}\ge\frac{9}{4}$$

http://www.ssmrmh.ro/2016/08/29/bagers-inequality-1/

212. $a,b,c\in\mathbb{C}^*$ **−different in pairs,** $|a|=|b|=|c|=1$,

$$a+b+c\in\{\pm 1,\pm i\}.\text{ Find: }\Omega=a^{-2021}+b^{-2021}+c^{-2021}$$

Proposed by Marian Ursărescu-Romania

Solution by Ravi Prakash-New Delhi-India

$a+b+c$ represents orthocentre of ΔABC

$a+b+c\in\{\pm 1,\pm i\}\Rightarrow a+b+c$ lies on the circumcircle of $\Delta ABC\Rightarrow\Delta ABC$ is right triangle.

If $a+b+c=\pm 1$, then one of the vertices is 1 or -1 and other two $i,-i$

$$\Omega=a^{-2021}+b^{-2021}+c^{-2021}=i^{-2021}+(-i)^{-2021}+(\pm i)^{2021}=\pm i$$

If $a+b+c=\pm i$, then one of the vertices is i or $-i$ and other two are $1,-1$.

$$\Omega=a^{-2021}+b^{-2021}+c^{-2021}=1^{-2021}+(-1)^{2021}+(\pm i)^{-2021}=\pm i$$

Therefore,

$$\Omega=a^{-2021}+b^{-2021}+c^{-2021}\in\{\pm 1,\pm i\}.$$

213. $a,b,c\in\mathbb{C}^*$ **−different in pairs,** $|a|=|b|=|c|$, $A(a),B(b),C(c)$. **Prove that:**

$$\sum_{cyc}\frac{|(a-b)(a-c)|}{|(a-b)|a-c|+(a-c)|a-b||^2}=9\left(\sum_{cyc}|a-b|\right)^{-2}\Leftrightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$A(a),B(b),C(c)\Rightarrow \Delta ABC\subset C(O,R),|a|=R$$

$$|a-b|=AB=c',|a-c|=AC=b',|b-c|=BC=a'\Rightarrow$$

$$\sum_{cyc}\frac{AB\cdot AC}{|(a-b)AC+(a-c)AB|^2}=\frac{9}{(AB+BC+CA)^2}\Leftrightarrow$$

$$\sum_{cyc}\frac{bc}{\left|\frac{(a-b)a'+(a-c)c'}{a+b+c}\right|^2}=9\Leftrightarrow\sum_{cyc}\frac{bc}{\left|\frac{(b+c)a'-bb'-cc'}{a+b+c}\right|^2}=9\Leftrightarrow$$

$$\sum_{cyc}\frac{bc}{\left|\frac{(a+b+c)a'-aa'-bb'-cc'}{a+b+c}\right|^2}=9\Leftrightarrow\sum_{cyc}\frac{bc}{\left|a'-\frac{aa'+bb'+cc'}{a+b+c}\right|^2}=9$$

$$\Rightarrow\sum_{cyc}\frac{bc}{AI^2}=9;(1)$$

But $\sum_{cyc}\frac{bc}{AI^2}=\frac{4R+r}{r}\geq 9\Leftrightarrow 4R+r\geq 9r\Leftrightarrow R\geq 2r;(2)$

From $(1)\&(2)$ we have equality, then ΔABC equilateral.

214. $a,b,c\in\mathbb{C}^*$ −different in pairs, $|a|=|b|=|c|,A(a),B(b),C(c)$. Prove that:

$$\sum_{cyc}\left|\frac{(a+b)(a+c)}{(a-b)(a-c)}\right|=1\Leftrightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$\sum_{cyc}\left|\frac{(a+b)(a+c)}{(a-b)(a-c)}\right|=1\Leftrightarrow\sum_{cyc}\frac{BH\cdot CH}{AB\cdot AC}=1\Leftrightarrow\sum_{cyc}\frac{BH\cdot CH}{BC}=1\Leftrightarrow$$

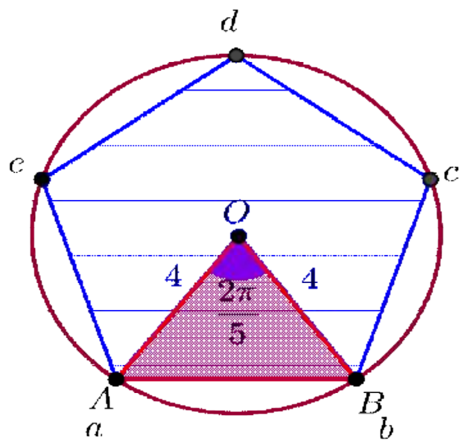
$$\sum_{cyc}aBH\cdot CH=abc;(1)$$

But $\sum_{cyc}aBH\cdot CH\geq abc;(2)$ true from

$$aPB\cdot PC+bPC\cdot PA+cPA\cdot PB\geq abc,\forall P\in\mathcal{P}$$

From $(1)\&(2)$ it following that ΔABC is equilateral.

215.



$a, b, c, d, e \in \mathbb{C}^*$ **–different in pairs,** $a + b + c + d + e = 0$,

$A(a), B(b), C(c), D(d), E(e)$,

$a^2 + b^2 + c^2 + d^2 + e^2 = 0, |a| = |b| = |c| = |d| = |e| = 4$. **Find:**

$$\Omega = \frac{[ABCDE]}{AB + BC + CD + DE + EA}$$

Proposed by Daniel Sitaru-Romania

Solution by Ravi Prakash-New Delhi-India

$$a + b + c + d + e = 0 \Rightarrow \sum_{cyc} a = 0 \Rightarrow \sum_{cyc} \bar{a} = 0 \Rightarrow \sum_{cyc} \frac{16}{a} = 0$$

$$\Rightarrow \sum_{cyc} abcd = 0$$

Next, $a^2 + b^2 + c^2 + d^2 + e^2 = 0$ **and** $a + b + c + d + e = 0$**, thus**

$$2\sum_{cyc} ab = \left(\sum_{cyc} a\right)^2 - \sum_{cyc} a^2 \Rightarrow \sum_{cyc} ab = 0 \Rightarrow \sum_{cyc} \bar{a}\bar{b} = 0 \Rightarrow$$

$$256 \sum_{cyc} \frac{1}{ab} = 0 \Rightarrow \sum_{cyc} abc = 0$$

Let us denote $abcde = \alpha$**, consider the equation whose roots are** a, b, c, d **and** e**, respectively.**

$$z^5-\left(\sum_{cyc}a\right)z^4+\left(\sum_{cyc}ab\right)z^3-\left(\sum_{cyc}abc\right)z-abcde=0$$

Or $z^5-\alpha=0;(1)$

Let one of the roots of (1) be β.

Then, roots of (1) are $\beta,\beta\omega,\beta\omega^2,\beta\omega^3,\beta\omega^4$, where $\omega=cos\frac{2\pi}{5}+isin\frac{2\pi}{5}$.

Let $a=\beta,b=\beta\omega,c=\beta\omega^2,d=\beta\omega^3,e=\beta\omega^4$,

$$[ABCDE]=5[AOB]=5\cdot 4^2\cdot sin\frac{2\pi}{5}.$$

Also, $AB=BC=CD=DE=EA$

$$AB=|b-a|=|\beta\omega-\beta|=|\beta||\omega-1|=|\beta||\omega||\omega-1|=|\beta||\omega^2-\omega|=BC\dots$$

Now, by the low of cosines:

$$AB^2=OA^2+OB^2-2OA\cdot OBcos\frac{2\pi}{5}=16+16-2\cdot 16\cdot 16cos\frac{2\pi}{5}=$$

$$=32\left[1-cos\left(\frac{2\pi}{5}\right)\right]=64sin^2\left(\frac{\pi}{5}\right)\Rightarrow AB=8sin\left(\frac{\pi}{5}\right)$$

$$\Omega=\frac{[ABCDE]}{AB+BC+CD+DE+EA}$$

$$=\frac{5\cdot 16\cdot sin\left(\frac{2\pi}{5}\right)}{5\cdot 8\cdot sin\left(\frac{\pi}{5}\right)}=\frac{4\cdot sin\left(\frac{2\pi}{5}\right)}{sin\left(\frac{\pi}{5}\right)}=\sqrt{5}+1$$

216. If $a,b,c\in\mathbb{C}^*$ –different in pairs, $|a|=|b|=|c|$, $A(a),B(b),C(c)$. Prove that:

$$\left(\sum_{cyc}\left((a-b)|a-c|+(a-c)|a-b|\right)\right)^2=\left(\sum_{cyc}|a-b|\right)^2\cdot\sum_{cyc}|a-b|^2\Rightarrow AB=BC=CA$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$|a-b|=AB=c_1,|b-c|=BC=a_1,|c-a|=CA=b_1$$

$$\left(\sum_{cyc}\left((a-b)|a-c|+(a-c)|a-b|\right)\right)^2=\left(\sum_{cyc}\left((a-b)b_1+(a-c)c_1\right)\right)^2=$$

$$=(a_1+b_1+c_1)^2(a_1^2+b_1^2+c_1^2)\Leftrightarrow$$

$$\left(\sum_{cyc}\left|\frac{b_1a-b_1b+c_1a-c_1c}{a_1+b_1+c_1}\right|\right)^2=a_1^2+b_1^2+c_1^2\Leftrightarrow$$

$$\left(\sum_{cyc}\left|\frac{(b_1+c_1)a-b_1b-c_1c}{a_1+b_1+c_1}\right|\right)^2=a_1^2+b_1^2+c_1^2\Leftrightarrow$$

$$\left(\sum_{cyc}\left|a-\frac{a_1a+b_1b+c_1c}{a_1+b_1+c_1}\right|\right)^2=a_1^2+b_1^2+c_1^2\Leftrightarrow$$

$$\left(\sum_{cyc}AI\right)^2=a_1^2+b_1^2+c_1^2;(1)$$

$$\text{But}\left(\sum_{cyc}AI\right)^2\le a_1^2+b_1^2+c_1^2;(2)$$

From (1),(2) equality holds if and only if triangle ABC is equilateral.

$$AI=\frac{r}{\sin\frac{A}{2}}=\sqrt{\frac{b_1c_1(s-a_1)}{s}}\Rightarrow(2)\Leftrightarrow\left(\sum_{cyc}\sqrt{\frac{b_1c_1(s-a_1)}{s}}\right)^2\le a_1^2+b_1^2+c_1^2;(3)$$

From BCS inequality, we have:

$$\left(\sum_{cyc}\sqrt{\frac{b_1c_1(s-a_1)}{s}}\right)^2\le\left(\sum_{cyc}b_1c_1\right)\left(\sum_{cyc}\frac{s-a_1}{s}\right)=\sum_{cyc}b_1c_1\le\sum_{cyc}a_1^2$$

$\Rightarrow(3)$ is true.

Solution 2 by George Florin Şerban-Romania

$BC=a',AC=b',AB=c',I$ –incenter.

$$AI=\left|\frac{aa'+bb'+cc'}{a'+b'+c'}-a\right|=\frac{|aa'+bb'+cc'-aa'-ab'-ac'|}{|a'+b'+c'|}=$$

$$= \frac{|b'(b-a)+c'(c-a)|}{2s} = \frac{|b'(a-b)+c'(a-c)|}{2s} \Rightarrow$$

$$2s\cdot AI = |b'(a-b)+c'(a-c)| = \big|(a-b)|a-c|+(a-c)|a-b|\big|$$

$$\sum_{cyc}\big|(a-b)|a-c|+(a-c)\{a-b|\big| = 2s\sum_{cyc}AI$$

$$\left(\sum_{cyc}\big|(a-b)|a-c|+(a-c)\{a-b|\big|\right)^2 = 4s^2\left(\sum_{cyc}AI\right)^2 =$$

$$= 4s^2\left(\sum_{cyc}AI^2+2\sum_{cyc}AI\cdot BI\right) = 4s^2\left(r^2\sum_{cyc}\frac{1}{sin^2\frac{A}{2}}+\frac{2abc}{s}\sum_{cyc}sin\frac{A}{2}\right) =$$

$$= 4s^2\left[r^2\frac{s^2+r^2-8Rr}{r^2}+\frac{8Rrs}{s}\sum_{cyc}sin\frac{A}{2}\right] =$$

$$= 4s^2\left(s^2+r^2-8Rr+8Rr\sum_{cyc}sin\frac{A}{2}\right) = \left(\sum_{cyc}|a-b|\right)^2\left(\sum_{cyc}|a-b|^2\right) =$$

$$= 4s^2\sum_{cyc}a'^2 = 4s^2(2s^2-2r^2-8Rr) = 2s^2-2r^2-8Rr$$

$$8Rr\sum_{cyc}sin\frac{A}{2} = s^2-3r^2 \Rightarrow \sum_{cyc}sin\frac{A}{2} = \frac{s^2-3r^2}{8Rr}$$

Let $f:(0,\pi)\to\mathbb{R}, f(x)=sin\frac{x}{2}, f''(x) = -\frac{1}{4}sin\frac{x}{2} < 0 \Rightarrow f$ –convex.

Applying Jensen inequality, we get:

$$\sum_{cyc}sin\frac{A}{2}\le\frac{3}{2}\Leftrightarrow\frac{s^2-3r^2}{8Rr}\le\frac{3}{2}\Leftrightarrow s^2\le 12Rr+3r^2$$

But $16Rr-5r^2\le s^2(Gerretsen)$. Remains to prove that:

$16Rr-5r^2\le 12Rr+3r^2\Leftrightarrow 4Rr\le 8r^2\Leftrightarrow R\le 2r$ but $R\ge 2r(Euler)$ then,

$R=2r\Leftrightarrow \Delta ABC$ equilateral.

217.

$$\frac{10x^8-480x^6+4032x^4-7680x^2+2560}{\frac{1024}{x^2}-11520+13440x^2-3360x^4+180x^6-x^8} = \frac{x}{2}$$

Proposed by Orlando Irahola Ortega-Bolivia

Solution by Lety Sauceda-Mexico City-Mexico

$$x^{10}+20x^9-180x^8-960x^7+3360x^6+8064x^5-13440x^4+15360x^3+11520x^2+5120x-1024=0$$

$$\frac{20x^9-960x^7+8064x^5-15360x^3+5120x}{1024-11529x^2+13440x^4-3369x^6+180x^8-x^{10}}=1$$

Dividing with 1024 and let $x=2w$ it follows that:

$$\frac{10w^9-120w^7+252w^5-120w^3+10w}{1-45w^2+210w^4-210w^6+45w^8-w^{10}}=1$$

Let $w=tan(a), x=2tan(a)$ then the up rapport is equivalent to $tan(10a)=1$.

$$tan(10a)=1\Rightarrow 10a=\frac{\pi}{4}\Rightarrow a=\frac{\pi}{40};\ x=2w=2tan\left(\frac{\pi}{40}(2k+1)\right)$$

So, $x=2tan\left(\frac{\pi}{40}(2k+1)\right), k=\{0,1,2,3,4,\dots\}$

218. If we have the relation

$$\sum_{n=0}^{\infty}\frac{\phi^{3n+1}+(-1)^n(3n+1)}{\phi^{4n+1}}=\frac{1+k}{k}\sum_{n=0}^{\infty}\frac{(-1)^n\left(n\phi+\sqrt{5}\right)+\phi^{2n+1}}{\phi^{3n+2}}$$

then find the value of k. (ϕ – Golden Ratio)

Proposed by Srinivasa Raghava-AIRMC-India

Solution by Izumi Ainsworth-Lima-Peru

$$\sum_{n=0}^{\infty}\frac{\phi^{3n+1}+(-1)^n(3n+1)}{\phi^{4n+1}}=\frac{1+k}{k}\sum_{n=0}^{\infty}\frac{(-1)^n\left(n\phi+\sqrt{5}\right)+\phi^{2n+1}}{\phi^{3n+2}}$$

$$\Rightarrow\frac{1}{\phi}\left[\phi\sum_{n=0}^{\infty}\left(\frac{1}{\phi}\right)^n+3\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{\phi^4}\right)^n}{n^{-1}}+\sum_{n=0}^{\infty}\left(-\frac{1}{\phi}\right)^n\right]=$$

$$=\frac{1+k}{k}\left[\frac{1}{\phi}\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{\phi^3}\right)^n}{n^{-1}}-\frac{\sqrt{5}}{\phi^2}\sum_{n=0}^{\infty}\left(-\frac{1}{\phi^3}\right)^n+\frac{1}{\phi}\sum_{n=0}^{\infty}\left(\frac{1}{\phi}\right)^n\right]$$

$$\Rightarrow\frac{1}{\phi}\left[\phi\left(\frac{1}{1-\frac{1}{\phi}}\right)+3Li_{-1}\left(-\frac{1}{\phi^4}\right)+\frac{1}{1-\frac{-1}{\phi^4}}\right]=$$

$$=\frac{1+k}{k}\left[\frac{1}{\phi}Li_{-1}\left(-\frac{1}{\phi^3}\right)+\frac{\sqrt{5}}{\phi^2}\left(\frac{1}{1-\frac{-1}{\phi^3}}\right)+\frac{1}{\phi}\left(\frac{1}{1-\frac{1}{\phi}}\right)\right]$$

$$\Rightarrow\frac{1}{\phi}\left[\phi\left(\frac{\phi}{\phi-1}\right)+3\left(\frac{\frac{-1}{\phi^4}}{\left(1-\frac{-1}{\phi^4}\right)^2}\right)+\frac{\phi^4}{\phi^4+1}\right]=$$

$$=\frac{1+k}{k}\left[\frac{1}{\phi}\left(\frac{-\frac{1}{\phi^3}}{\left(1-\frac{-1}{\phi^3}\right)^2}\right)+\frac{\sqrt{5}}{\phi^2}\left(\frac{\phi^3}{\phi^3+1}\right)+\frac{1}{\phi}\left(\frac{\phi}{\phi-1}\right)\right]$$

$$\Rightarrow\frac{1}{\phi}\left[\phi(\phi^2)-3\frac{\phi^4}{(\phi^4+1)^2}+\frac{\phi^4}{\phi^4+1}\right]=$$

$$=\frac{1+k}{k}\left[-\frac{1}{\phi}\left(\frac{\phi^3}{(\phi^3+1)^2}\right)+\frac{\sqrt{5}}{\phi^2}\left(\frac{\phi^3}{\phi^3+1}\right)+\frac{1}{\phi}(\phi^2)\right]$$

$$\Rightarrow\frac{1}{\phi}\left[\phi^3-3\frac{\phi^4}{(3\phi^2)^2}+\frac{\phi^4}{3\phi^2}\right]=\frac{1+k}{k}\left[-\frac{1}{\phi}\left(\frac{\phi^3}{(2\phi^2)^2}\right)+\frac{\sqrt{5}}{2}\left(\frac{\phi^3}{2\phi^2}\right)+\phi\right]$$

$$\Rightarrow\frac{1}{\phi}\left[(2\phi+1)-\frac{1}{3}+\frac{\phi+1}{3}\right]=\frac{1+k}{k}\left(-\frac{1}{4\phi^2}+\frac{\sqrt{5}}{2\phi}-\frac{1}{4}\right)$$

$$\frac{1}{3}(7\phi^2+3\phi)=\frac{1+k}{k}\left(\phi^3+\frac{\sqrt{5}}{2}\phi-\frac{1}{4}\right)$$

$$\frac{1}{3}(7(\phi+1)+3\phi)=\frac{1+k}{k}\left((2\phi+1)+\frac{\phi}{2}+1-\frac{1}{4}\right)$$

$$\frac{10\phi+7}{3}=\frac{1+k}{k}\left(\frac{10\phi+7}{4}\right)\Rightarrow\frac{4}{3}=\frac{1+k}{k}\Rightarrow k=3.$$

219. Solve for real numbers:

$$2x[x]+2\big[x[x]\big]=5;\ [*]-GIF$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Bedri Hajrizi-Mitrovica-Kosovo

$$x[x]+\big[x[x]\big]=2,5\Leftrightarrow\big[x[x]\big]=2,5-x[x]$$
$$2,5-x[x]\le x[x]<3,5-x[x]$$
$$1,25\le x[x]<1,75\Rightarrow\big[x[x]\big]=1$$
$$x[x]+1=2,5\Rightarrow x[x]=1,5\Rightarrow[x]=\frac{1,5}{x}$$

$$\frac{1,5}{x} \le x < \frac{1,5}{x} + 1$$

$$(I) - \sqrt{\frac{3}{2}} \le x < \frac{1-\sqrt{7}}{2} \Rightarrow x \in \left[-\sqrt{\frac{3}{2}}, \frac{1-\sqrt{7}}{2}\right) \Rightarrow [x] \in \{-2; -1\} \Rightarrow$$

$$x \in \{-0,75; -1,5\} - \text{no solution}.$$

$$(II) \sqrt{\frac{3}{2}} \le x < \frac{1+\sqrt{7}}{2} \Rightarrow [x] = 1, S = \{1\}$$

220.

$$A = \sum_{i=1}^{n} \int_{k+1}^{k+2} \frac{(x+1)dx}{x^4 + 4x^3 + (4i+2)x^2 + (8i-4)x + 4i^2 - 4i}$$

$$\Omega = \lim_{k\to\infty} Ak^3$$

Solve for natural numbers:

$$\binom{\Omega}{6} = \binom{n}{2014}$$

Proposed by Costel Florea-Romania

Solution 1 by Adrian Popa-Romania

$$I = \int_{k+1}^{k+2} \frac{(x+1)dx}{(x^2+2x+2n)\left(x^2+2x+2(n-1)\right)} =$$

$$= \frac{1}{2}\int_{k+1}^{k+2} \frac{(x+1)dx}{x^2+2x+2(n-1)} - \frac{1}{2}\int_{k+1}^{k+2} \frac{(x+1)dx}{x^2+2x+2n} = I_1 - I_2$$

$$I_1 = \frac{1}{2}\int_{k+1}^{k+2} \frac{(x+1)dx}{x^2+2x+2(n-1)} \overset{x^2+2x+2(n-1)=t}{=} \frac{1}{2}\int_{(k+2)^2+2(n-1)-1}^{(k+3)^2+2(n-1)-1} \frac{dt}{t} =$$

$$= \frac{1}{2}log\left(\frac{(k+3)^2+2(n-1)-1}{(k+2)^2+2(n-1)-1}\right)$$

$$I_2 = \frac{1}{2}\int_{k+1}^{k+2} \frac{(x+1)dx}{x^2+2x+2n} \overset{x^2+2x+2n=t}{=} \frac{1}{2}\int_{(k+2)^2+2n-1}^{(k+3)^2+2n-1} \frac{dt}{t} = \frac{1}{2}log\left(\frac{(k+3)^2+2n-1}{(k+2)^2+2n-1}\right)$$

Hence,

$$A=\frac{1}{4}\sum_{i=1}^{n}\left(log\left(\frac{(k+3)^2+2(i-1)-1}{(k+2)^2+2(i-1)-1}\right)-log\left(\frac{(k+3)^2+2i-1}{(k+2)^2+2i-1}\right)\right)=$$

$$=\frac{1}{4}log\left(1+\frac{4nk+10n}{k^4+10k^3+(2n+35)k^2+(8n+50)k+6n+24}\right)$$

On the other hand,

$$\Omega=\lim_{k\to\infty}Ak^3=\frac{1}{4}\lim_{k\to\infty}log\left(\underbrace{1+\frac{4nk+10n}{k^4+10k^3+(2n+35)k^2+(8n+50)k+6n+24}}_{E}\right)^k$$

$$\left(\lim_{k\to\infty}E=e^{\lim_{k\to\infty}\frac{(4nk+10n)k^3}{k^4+10k^3+(2n+35)k^2+(8n+50)k+6n+24}}=e^{4n}\right)$$

$$\Omega=\frac{1}{4}loge^{4n}=n$$

Therefore,

$$\binom{n}{6}=\binom{n}{2014}\Rightarrow\binom{n}{6}=\binom{n}{n-2014}\Rightarrow 6=n-2014\Rightarrow n=2020$$

Solution 2 by Ravi Prakash-New Delhi-India

$$x^4+4x^3+(4n+2)x^2+(8n-4)x+4n^2-4n=$$

$$=[(x+1)^2+2(n-1)]^2-1$$

$$I_n:=\int_{k+1}^{k+2}\frac{(x+1)dx}{x^4+4x^3+(4n+2)x^2+(8n-4)x+4n^2-4n}=$$

$$=\int_{k+1}^{k+2}\frac{(x+1)dx}{[(x+1)^2+2(n-1)]^2-1}\overset{(x+1)^2+2(n-1)=t}{=}$$

$$=\frac{1}{2}\int_{(k+2)^2+2(n-1)}^{(k+3)^2+2(n-1)}\frac{dt}{t^2-1}=-\frac{1}{4}log\left(\frac{(k+3)^2+2n-1}{(k+3)^2+2n-3}\right)+\frac{1}{4}log\left(\frac{(k+2)^2+2n-1}{(k+2)^2+2n-3}\right)$$

$$A=\sum_{r=1}^{n}I_r=\frac{1}{4}\sum_{r=1}^{n}\left[log\left(\frac{(k+3)^2+2r-1}{(k+3)^2+2r-3}\right)+log\left(\frac{(k+2)^2+2r-1}{(k+2)^2+2r-3}\right)\right]=$$

$$=-\frac{1}{4}log\left(\frac{(k+3)^2+2n-1}{(k+3)^2-1}\right)+\frac{1}{4}log\left(\frac{(k+2)^2+2n-1}{(k+2)^2-1}\right)=$$

$$= \frac{1}{4}log\left(1+\frac{2n}{(k+2)^2-1}\right)-\frac{1}{4}log\left(1+\frac{2n}{(k+3)^2-1}\right) =$$

$$= \frac{1}{4}\left[\frac{2n}{(k+2)^2-1}-\frac{2n}{(k+3)^2-1}+\frac{1}{2}\left(\frac{2n}{(k+2)^2-1}\right)^2-\frac{1}{2}\left(\frac{2n}{(k+3)^2-1}\right)^2+\cdots\right] =$$

$$= \frac{n}{2}\left[\frac{(k+3+k+2)}{((k+2)^2-1)((k+3)^2-1)}+o\left(\frac{1}{k^4}\right)\right]$$

Hence,

$$\Omega = \lim_{k\to\infty} Ak^3 = \frac{n}{2}\lim_{k\to\infty}\left(\frac{2k^4+5k^3}{((k+2)-1)((k+3)^2-1)}\right) = n$$

Therefore,

$$\binom{n}{6} = \binom{n}{2014} \Rightarrow n = 2020$$

221. Find all real numbers $\alpha \geq 0$ such that:

$$sin^2(\sqrt{\alpha}x^{2020}) + cos^2\left((\alpha^2-1)x^{2020}\right) = 1, \forall x \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$sin^2(\sqrt{\alpha}x^{2020}) + cos^2\left((\alpha^2-1)x^{2020}\right) = 1, ;\ (*)$ **true for all $x \in \mathbb{R}$ if and only if:**

$$\sqrt{\alpha}x^{2020} = (\alpha^2-1)x^{2020}, \forall x \in \mathbb{R} \Leftrightarrow$$

$$(\alpha^2-\sqrt{\alpha}-1)x^{2020} = 0, \forall x \in \mathbb{R} \Leftrightarrow$$

$$\alpha^2-\sqrt{\alpha}-1 = 0 \Leftrightarrow \alpha^2 = 1+\sqrt{\alpha}$$

So, for $\alpha \geq 0$ such that $\alpha^2 = 1+\sqrt{\alpha}$ $(\alpha \approx 1,2207) \Rightarrow (*)$ is true for $x \in \mathbb{R}$.

222. Solve for real numbers:

$$(x-[x])^{x-[x]} + (x+[x])^{x+[x]} = 4; [*] - GIF$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Michael Sterghiou-Greece

$$(x-[x])^{x-[x]} + (x+[x])^{x+[x]} = 4;\ \ (T)$$

In the below we treat on $\mathbb{R}$ only integer powers of negative numbers.
In general $a^b \in \mathbb{C}$ when $a < 0$ and $b \notin \mathbb{Z}^*$

Also 0^0 can be undefined or be assigned the value of 1.

Now: 1) if $0 < x < 1$ then $[x] = 0$ and $(T) \Rightarrow \{x\}^{\{x\}} + x^x = 4$

$0 < \{x\} = x - [x] < 1$ so $\{x\}^{\{x\}} < 1$ and $x^x < 1$, no solutions.

2) if $1 < x$ then $x + [x] > 2$ as both $[x] \geq 1, x > 1$ therefore,

$(x + [x])^{(x+[x])} > 2^2 = 4$ and $(x - [x])^{(x-[x])} > 0$ No solutions.

3) if $x = 0$ then (T) is undefined or reduces to $1 + 1 = 4$.

4) if $x = 1$ then (T) is undefined $(x - [x] = 0)$ or reduces to $1 + 4 = 4$ (false).

5) if $x \in \mathbb{Z}_-$ then $x - [x] = 0$ and $x + [x] = -2|x|$ so (T) either undefined or reduces to

$4 = 1 + \frac{1}{4^{|x|}}\left(\frac{1}{|x|}\right)^{2|x|} < 2$ as $|x| \geq 1$ and $\frac{1}{4^{|x|}}\left(\frac{1}{|x|}\right)^{2|x|} \leq \frac{1}{4}$, contradiction.

Therefore, with the above assumptions set of solutions $= \emptyset$

223. Solve for real numbers:

$$[x^2] + x^2 = 2x[x], \quad [*] - GIF$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

Let $x = k + a, k \in \mathbb{Z}, a \in [0, 1)$

$$[k^2 + 2ka + a^2] + k^2 + 2ka + a^2 = 2(k + a)k$$

$$2k^2 + [2ka + a^2] + 2ka + a^2 = 2ka + ak^2$$

For $a = 0$, we get solution $x = k$.

For $a \neq 0$: $[2ka + a^2] + a^2 = 0$

Being that $a \in (0, 1) \Rightarrow a^2 \in (0, 1)$, so $[2ka + a^2] = -a^2 \in (-1, 0)$ which is impossible.

Solution 2 by Rachid Iksi-Morocco

$$[x^2] + x^2 = 2x[x]$$

$x = [x]$ is a solution for the equation.

$$1)\ [x] \geq 0 \Rightarrow [x]^2 \leq x^2 \Rightarrow [x]^2 \leq [x^2] \Rightarrow$$

$$[x]^2 + x^2 \leq [x^2] + x^2 = 2x[x] \Rightarrow x^2 + [x]^2 - 2x[x] \leq 0 \Rightarrow$$

$$(x - [x])^2 \leq 0 \Rightarrow x = [x]$$

$$2)\ [x] \leq -1 \Rightarrow [x]^2 \geq x^2 \geq [x^2] \Rightarrow$$

$$[x]^2 + x^2 \geq 2x^2 \geq [x^2] + x^2 = 2x[x]$$

$\Rightarrow x^2 \geq x[x] \Rightarrow x \leq [x]$ as $[x] \leq -1, x < 0 \Rightarrow x = [x]$ is the solution.

So, $x = [x]$ is the only solution. Set is $\mathbb{Z}$.

224. **Solve for real numbers:**

$$[x^2] + x^2 = 2x[x], \quad [*] - GIF$$

Proposed by Jalil Hajimir-Toronto-Canada

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

Let $x = k + a, k \in \mathbb{Z}, a \in [0, 1)$

$$[k^2 + 2ka + a^2] + k^2 + 2ka + a^2 = 2(k + a)k$$

$$2k^2 + [2ka + a^2] + 2ka + a^2 = 2ka + ak^2$$

For $a = 0$, we get solution $x = k$.

For $a \neq 0$: $[2ka + a^2] + a^2 = 0$

Being that $a \in (0, 1) \Rightarrow a^2 \in (0, 1)$, so $[2ka + a^2] = -a^2 \in (-1, 0)$ which is impossible.

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$$[x^2] + x^2 = 2x[x]$$

$x = [x]$ is a solution for the equation.

$$1)\ [x] \geq 0 \Rightarrow [x]^2 \leq x^2 \Rightarrow [x]^2 \leq [x^2] \Rightarrow$$

$$[x]^2 + x^2 \leq [x^2] + x^2 = 2x[x] \Rightarrow x^2 + [x]^2 - 2x[x] \leq 0 \Rightarrow$$

$$(x - [x])^2 \leq 0 \Rightarrow x = [x]$$

$$2)\ [x] \leq -1 \Rightarrow [x]^2 \geq x^2 \geq [x^2] \Rightarrow [x]^2 + x^2 \geq 2x^2 \geq [x^2] + x^2 = 2x[x]$$

$\Rightarrow x^2 \geq x[x] \Rightarrow x \leq [x]$ as $[x] \leq -1, x < 0 \Rightarrow x = [x]$ is the solution.

So, $x = [x]$ is the only solution. Set is $\mathbb{Z}$.

225. **Solve for real numbers:**

$$2 + x^2 + y^2 + z^2 + t^2 = xy + yz + zt + tx + 2|x - y + z - t|$$

Proposed by Mihály Bencze-Romania

Solution by Chris Kyriazis-Greece

$$4 + 2x^2 + 2y^2 + 2z^2 + 2t^2 = 2(x + z)(y + t) + 4|(x + z) - (y + t)| \Leftrightarrow$$

$$(x + z)^2 + (y + t)^2 - 2(x + z)(y + t) - 4|(x + z) - (y + t)| + 4 + x^2 + y^2 + z^2 + t^2 - 2xz - 2yt = 0 \Leftrightarrow$$

$$|(x + z) - (y + t)|^2 - 4|(x + z) - (y + t)| + 4 + (x - 2)^2 + (y - t)^2 = 0 \Leftrightarrow$$

$$(|(x + z) - (y + t)| - 2)^2 + (x - z)^2 + (y - t)^2 = 0$$

This means:

$$\begin{cases} x = z \\ y = t \\ |x + z - (y + t)| = 2 \end{cases} \Leftrightarrow \begin{cases} x = z \\ y = t \\ x - y = 1 \end{cases} \text{ or } \begin{cases} x = z \\ y = t \\ x - y = -1 \end{cases}$$

Case I) If $x = 2, y = t, x - y = 1$, we deduce that the solution is:

$$\{(x, x-1, x, x-1) \mid x \in \mathbb{R}\}$$

Case II) If $x = z, y = t, x - y = -1$, we deduce that the solution is:

$$\{(x, x+1, x, x+1) \mid x \in \mathbb{R}\}$$

226. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \sum_{cyc} \frac{xy}{\sqrt{(1+x^2)(1+y^2)}} = \frac{3}{2} \\ xy + yz + zx = 3 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by George Florin Şerban-Romania

$$\frac{3}{2} = \sum_{cyc} \frac{xy}{\sqrt{(1+x^2)(1+y^2)}} \overset{CBS}{\leq} \sum_{cyc} \frac{xy}{xy+1} \Rightarrow \sum_{cyc} \frac{xy}{xy+1} \geq \frac{3}{2}$$

$$\Leftrightarrow \sum_{cyc} \frac{xy - 1 + 1}{xy+1} \geq \frac{3}{2} \Leftrightarrow \sum_{cyc} \left(1 - \frac{1}{xy+1}\right) \geq \frac{3}{2} \Leftrightarrow \sum_{cyc} \frac{1}{xy+1} \leq \frac{3}{2}$$

$$\sum_{cyc} \frac{1}{xy+1} \overset{Bergstrom}{\geq} \frac{9}{(xy+yz+zx+3} = \frac{3}{2} \Rightarrow xy = yz = zx = 1$$

$$(xyz)^2 = 1, xyz > 0 \Rightarrow xyz = 1 \Rightarrow x = y = z = 1.$$

227. Solve for real numbers:

$$\begin{cases} x[y] + y[z] + z[x] = 11 \\ x[z] + y[x] + z[y] = 11 \\ x[x] + y[y] = z[z] = 14 \end{cases}$$

$[*]$ —is the greatest integer part of $*$

Proposed by Jalil Hajimir-Toronto-Canada

Solution by Santos Martins Junior-Brusels-Belgium

Adding up equations of the system yields:

$$(x+y+z)([x]+[y]+[z])=36$$

$$([x]+[y]+[z]+\{x\}+\{y\}+\{z\})([x]+[y]+[z])=36$$

$$([x]+[y]+[z])^2+(\{x\}+\{y\}+\{z\})([x]+[y]+[z])-36=0$$

Quadratic in $([x]+[y]+[z])$

whose discriminant D must be a perfect square ≥ 0

$$D=(\{x\}+\{y\}+\{z\})^2+144$$

We know that by definition $0\leq\{r\}<1$ **for any real number** r

Hence $0\leq\{x\}+\{y\}+\{z\}<3$

implying that D is a perfect square only for $\{x\}+\{y\}+\{z\}=0$

$\Rightarrow\{x\}=\{y\}=\{z\}=0\Rightarrow x,y,z$ **are all integers.**

Hencesystem becomes: $xy+yz+zx=11$; **(1) and** $x^2+y^2+z^2=14$ **;(2) where** x,y,z **are all integers**

From (2) we easily get that the triplet $(x,y,z)=(1,2,3)$ **and its permutations and the triplet** $(x,y,z)=(-1,-2,-3)$ **and its permutations are solutions of the system**

228. Solve for real numbers:

$$\begin{cases}\sqrt[3]{(x+3)^2}+6\sqrt[3]{(z-3)^2}=5\sqrt[3]{(x-3)(y+3)}\\ x,y,z>0\\ \dfrac{2x^2}{yz(y+z)}+\dfrac{2y^2}{zx(z+x)}+\dfrac{2z^2}{xy(x+y)}=\dfrac{9}{x+y+z}\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Rahim Shahbazov-Baku-Azerbaijan

$$\frac{x^3}{a}+\frac{y^3}{b}+\frac{z^3}{c}\geq\frac{(x+y+z)^3}{3(a+b+c)}\ (Holder)$$

$$\sum_{cyc}\frac{2x^2}{yz(y+z)}=\frac{2}{xyz}\sum_{cyc}\frac{x^3}{y+z}\geq\frac{2}{xyz}\cdot\frac{(x+y+z)^3}{6(x+y+z)}=$$

$$=\frac{(x+y+z)^2}{3xyz}\geq\frac{9}{x+y+z}\Rightarrow(x+y+z)^3\overset{AGM}{\geq}27xyz\Rightarrow x=y=z\Rightarrow$$

$$\sqrt[3]{(x+3)^2}+6\sqrt[3]{(x-3)^2}=5\sqrt[3]{(x-3)(x+3)}$$

Let $t=\sqrt[3]{\frac{x+3}{x-3}}\Rightarrow t^2-5t+6=0\Rightarrow t_1=2;t_2=3$

Hence,

$$\sqrt[3]{\frac{x+3}{x-3}}=2\Rightarrow x=\frac{27}{7},\ \sqrt[3]{\frac{x+3}{x-3}}=3\Rightarrow x=\frac{42}{13}$$

Therefore $(x,y,z)=\left\{\left(\frac{27}{7},\frac{27}{7},\frac{27}{7}\right);\left(\frac{42}{13},\frac{42}{13},\frac{42}{13}\right)\right\}$

229. Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ 32(x^5+y^5+1)=(x+y)^5+(x+1)^5+(y+1)^5 \\ x+z=\sqrt{xz}+\sqrt{\dfrac{x^2+z^2}{2}} \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Alex Szoros-Romania

$$f:(0,\infty)\to\mathbb{R},f(x)=x^5\Rightarrow f'(x)=5x^4,f''(x)=20x^3\Rightarrow\frac{f(x)+f(y)}{2}\geq f\left(\frac{x+y}{2}\right)$$

$$\Rightarrow\frac{x^5+y^5}{2}\geq\left(\frac{x+y}{2}\right)^5;\frac{x^5+1}{2}\geq\left(\frac{x+1}{2}\right)^5;\frac{y^5+1}{2}\geq\left(\frac{y+1}{2}\right)^5\Rightarrow$$

$$\frac{x^5+y^5}{2}+\frac{x^5+1}{2}+\frac{y^5+1}{2}\geq\left(\frac{x+y}{2}\right)^5+\left(\frac{x+1}{2}\right)^5+\left(\frac{y+1}{2}\right)^5$$

$$32(x^5+y^5+1)\geq(x+y)^5+(x+1)^5+(y+1)^5;\forall x,y>0$$

$$32(x^5+y^5+1)=(x+y)^5+(x+1)^5+(y+1)^5\Leftrightarrow x=y=1$$

$$x=1\Rightarrow 1+z=\sqrt{z}+\sqrt{\frac{1+z^2}{2}}$$

$$1+2z+z^2=z+2\sqrt{\frac{z(1+z^2)}{2}}+\frac{1+z^2}{2}$$

$$\Rightarrow \frac{z^2+1}{2}+z=2\sqrt{\frac{z(1+z^2)}{2}}$$

$$(z+1)^2=2\sqrt{2z(z^2+1)} \Leftrightarrow (z+1)^4=8z(z^2+1) \Leftrightarrow$$

$z^4-4z^3+6z^2-4z+1=0 \Leftrightarrow x=y=z=1$ solution.

Solution 2 by Ravi Prakash-New Delhi-India

For $x, y>0$ we have:

$$16(x^5+y^5) \geq (x+y)^5 \Leftrightarrow 15(x^5+y^5) \geq 5x^4y+10x^3y+10x^2y^3+5xy^4$$

$$\Leftrightarrow (x^5-x^4y)+2(x^5-x^3y^2)+2(x^5-x^2y^3)+(y^5-xy^4) \geq 0$$

$$\Leftrightarrow x^4(x-y)+2x^3(x^2-y^2)-2y^3(x^2-y^2)-y^4(x-y) \geq 0$$

$$\Leftrightarrow (x^4-y^4)(x-y)+2(x^3-y^3)(x^2-y^2) \geq 0$$

Thus, $16(x^5+y^5) \geq (x+y)^5$

Equality holds when $x=y$.

Now, $32(x^5+y^5+1)=(x+y)^5+(x+1)^5+(y+1)^5$

$$[16(x^5+y^5)-(x+y)^5]+[16(x^5+1)-(x+1)^5]+[16(y^5+1)-(y+1)^5]=0$$

$$\Leftrightarrow 16(x^5+y^5)=(x+y)^5, 16(x^5+1)=(x+1)^5, 16(y^5+1)=(y+1)^5$$

$\Leftrightarrow x=y=1$. Next,

$$x+z=\sqrt{xz}+\frac{1}{\sqrt{2}}\sqrt{x^2+z^2}; \quad (2)$$

Put: $x=r\cos\theta, z=r\sin\theta$, hence (2) becomes

$$\cos\theta+\sin\theta=\sqrt{\sin\theta\cos\theta}+\frac{1}{\sqrt{2}} \Leftrightarrow \sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\sqrt{\sin(2\theta)}+\frac{1}{\sqrt{2}}$$

Put $\theta-\frac{\pi}{4}=\varphi$ so that $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$, hence

$$2\cos\varphi=\sqrt{\cos(2\varphi)}+1$$

$$\Leftrightarrow 4\cos^2\varphi-4\cos\varphi+1=2\cos^2\varphi-1$$

$$\Leftrightarrow 2\cos^2\varphi-4\cos\varphi+2=0$$

$$\Leftrightarrow (cos\varphi - 1)^2 = 0 \Leftrightarrow \varphi = 0 \Leftrightarrow \theta = \frac{\pi}{4}, \text{ thus } z = x = 1.$$

Therefore, $x = y = z = 1$.

Solution 3 by Abdul Hannan-Tezpur-India

$$\forall a, b > 0, 16(a^5 + b^5) = 32\left(\frac{a^5 + b^5}{2}\right) \overset{power-mean}{\geq} 32\left(\frac{a+b}{2}\right)^5 = (a+b)^5;\ (*)$$

Equality holdes if and only if $a = b$.

$$\Rightarrow 32(x^5 + y^5 + 1) = 16(x^5 + y^5) + 16(x^5 + 1^5) + (y^5 + 1^5) \overset{(*)}{\geq}$$

$$\geq (x+y)^5 + (x+1)^5 + (y+1)^5$$

Since equality holds, $x = y, x = 1, y = 1$ which implies that $x = y = 1$.

On the other hand, $\frac{x^2+z^2}{2} + xz \overset{AGM}{\geq} 2\sqrt{xz\left(\frac{x^2+z^2}{2}\right)}$ with equality iff $\frac{x^2+z^2}{2} = xz \Leftrightarrow$

$$(x - z)^2 = 0 \Leftrightarrow x = z$$

$$\Rightarrow (x+z)^2 = \frac{x^2+z^2}{2} + xz + \frac{x^2+z^2}{2} + xz \geq \frac{x^2+z^2}{2} + xz + 2\sqrt{xz\left(\frac{x^2+z^2}{2}\right)} =$$

$$= \left(\sqrt{\frac{x^2+z^2}{2}} + \sqrt{xz}\right)^2 \Rightarrow x + z \geq \sqrt{\frac{x^2+z^2}{2}} + \sqrt{xz}$$

Since equality holds, $z = x$. Therefore, $x = y = z = 1$.

230. Solve for real numbers:

$$\begin{cases} \dfrac{x^5}{y} + x^2y^2 + \dfrac{y^5}{x} = \sqrt{3(x^8 + y^8 + x^4y^4)} \\ \dfrac{3xy + y^4}{1 + 3y^2} = \dfrac{x}{\pi} \end{cases}$$

Proposed by Orlando Irahola Ortega-Tarija-Bolivia

Solution by Rahim Shahbazov-Baku-Azerbaijan

$$\frac{x^5}{y} + x^2y^2 + \frac{y^5}{x} = \frac{x^6 + x^3y^3 + y^6}{xy}; x^6 + x^3y^3 + y^6 > 0 \Rightarrow xy > 0$$

If $x, y < 0$ then second question is not true, so we have $x, y > 0$.

$$(x^6+x^3y^3+y^6)^2=3x^2y^2(x^8+x^4y^4+y^8);\ \ (1)$$

$$A=x^2+y^2, B=xy\overset{(1)}{\Rightarrow}(A^3-3AB^2+B^3)^2=3B^2(A^4-4A^2B^2+3B^4)$$

$$(t^3-3t+1)^2=3(t^4-4t^2+3), \text{where } t=\frac{A}{B}\geq 2$$

$$\Rightarrow t^6-9t^4+2t^3+21t^2-6t-8=0$$

$$\Rightarrow (t-2)\left(t^5+2t^4-5t^3-8t^2+5t+4\right)=0, \text{ true because}$$

$$t^5+2t^4-5t^3-8t^2+5t-4=(t-2)(t^4+4t^3+3t^2-2t+1)+6>0$$

$$t=2\Rightarrow x^2+y^2=2xy\Rightarrow x=y \text{ then, we get:}$$

$$\frac{3x^2+x^4}{1+3x^2}=\frac{x}{\pi}\Rightarrow\frac{3x+x^3}{1+3x^2}=\frac{1}{\pi}; x\neq 0$$

$$\text{Let } x=tanhx\Rightarrow tanh(3x)=\frac{1}{\pi}\Rightarrow\frac{e^{3u}-e^{-3u}}{e^{3u}+e^{-3u}}=\frac{1}{\pi}$$

$$\Rightarrow\frac{e^{6u}-1}{e^{6u}+1}=\frac{1}{\pi}\Rightarrow e^{6u}=\frac{\pi+1}{\pi-1}\Rightarrow u=\frac{1}{6}log\left(\frac{\pi+1}{\pi-1}\right)$$

Therefore,

$$x=y=tan\,u; u=\frac{1}{6}log\left(\frac{\pi+1}{\pi-1}\right)$$

231. Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ tan(1024x)+tan(1024y)+tan(1024z)=0 \\ x+y+z=\pi \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Mohammad Rostami-Kabul-Afganistan

$$tan(1024x)+tan(1024y)=-tan(1024z);\ \ (I)$$

$$x+y+z=\pi\Rightarrow x+y=\pi-z\Rightarrow 1024x+1024y=1024\pi-1024z$$

$$\Rightarrow tan(1024x+1024y)=tan(1024\pi-1024z)$$

$$\Rightarrow\frac{tan1024x+tan1024y}{1-tan1024x\cdot tan1024y}=-tan1024z$$

$$tan1024x+tan1024y=$$

$$=tan1024x+tan1024y-tan1024x\cdot tan1024y(tan1024x+tan1024y)$$

$$\Leftrightarrow tan1024x\cdot tan1024y(tan1024x+tan1024y)=0$$

$$\Rightarrow \begin{cases} tan1024x \cdot tan1024y = 0; & (II) \\ tan1024x + tan1024y; & (III) \end{cases}$$

$$(II): \begin{cases} tan1024x = 0 \\ tan1024y = 0 \end{cases} \Rightarrow \begin{cases} x = \dfrac{k\pi}{1024} \\ y = \dfrac{k'\pi}{1024} \end{cases}; k, k' \in \mathbb{Z}$$

$$(I)\&(III): tan1024z = 0 \Rightarrow z = \frac{k''\pi}{1024}; k'' \in \mathbb{Z}$$

If $k = k' = 0 \Rightarrow x = y = 0, k'' = 1024 \Rightarrow \begin{cases} x = y = 0 \\ z = \pi \end{cases}$

If $k' = k'' = 0 \Rightarrow y = z = 0, x = \pi, k = 1024 \Rightarrow \begin{cases} y = z = 0 \\ x = \pi \end{cases}$

If $k = k'' = 0 \Rightarrow x = z = 0, y = \pi, k' = 1024 \Rightarrow \begin{cases} x = z = 0 \\ y = \pi \end{cases}$

$$x + y + z = \pi \Rightarrow \frac{k\pi}{1024} + \frac{k'\pi}{1024} + \frac{k''\pi}{1024} = \pi \Rightarrow \frac{\pi}{1024}(k + k' + k'') = \pi$$

$$k + k' + k'' = 1024$$

$$x = \frac{k\pi}{1024}; y = \frac{k'\pi}{1024}; z = \frac{k''}{1024}; x + y + z = \pi' k, k', k'' \in \mathbb{Z}$$

232. If $A \in M_3(\mathbb{R})$ such that $det(A^2 - 3A + 3I_3) = 0$. Prove that:

$$2det(A^2 + 3I_3) \geq 3(detA + 3)^2$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$det(A^2 - 3A + 3I_3) = det(A + \alpha I_3)det(A - \alpha I_3) = 0; (1)$, where $\alpha \in \mathbb{C}$ is root of

equation $x^2 + 3x + 3 = 0, \alpha = \frac{-3 \pm i\sqrt{3}}{2}, \alpha + \overline{\alpha} = -3, \alpha \cdot \overline{\alpha} = 3 \overset{(1)}{\Rightarrow}$

$det(A + \alpha I_3) = 0$ or $det(A - \alpha I_3) = 0; det(\overline{A + \alpha I_3}) = 0; (2)$

Let $f(x) = det(A + xI_3) = detA + a_1 x + a_2 x^2 + x^3 \overset{(2)}{\Rightarrow} f(\alpha) = 0$

$\Rightarrow detA + a_1\alpha + a_2\alpha^2 + \alpha^3 = 0; (3)$

But $\alpha = -1 + \varepsilon$, where $\varepsilon^2 + \varepsilon + 1 = 0$ or $\varepsilon^3 = 1; (4)$.

From (3),(4) we have: $detA + a_1(\varepsilon - 1) + a_2(\varepsilon - 1)^2 + (\varepsilon - 1)^3 = 0 \Rightarrow$

$$detA - a_1 + a_1\varepsilon - 3\varepsilon a_2 + 3(2\varepsilon + 1) = detA - a_1 + 3 + \varepsilon(a_1 - 3a_2 + 6) = 0$$

$$a_1 = detA + 3, a_2 = \frac{1}{3}(detA + 9) \Rightarrow$$

$$f(x) = detA + (3 + detA)x + \frac{1}{3}(9 + detA)x^2 + x^3$$

$$f(i\sqrt{3}) = det(A + i\sqrt{3}) = detA + (3 + detA)i\sqrt{3} - detA - 9 - i3\sqrt{3} =$$

$$= -9 + i\sqrt{3}detA = -(9 - i\sqrt{3}detA)$$

$$f(-i\sqrt{3}) = det(A - i\sqrt{3}) = -(9 + i\sqrt{3}detA)$$

$$f(i\sqrt{3})f(-i\sqrt{3}) = det(A + 3I_3) = 81 + 3(detA)^2$$

$$= 3(9 + (detA)^2) \overset{BCS}{\geq} \frac{3}{2}(detA + 3)^2$$

233. If $x_1, x_2, \dots, x_{2021}$ are positive real numbers such that:

$$\frac{1}{1 + x_1} + \frac{1}{1 + x_2} + \dots + \frac{1}{1 + x_{2021}} = \frac{1}{2020}$$

Prove that: $x_1 \cdot x_2 \cdot \dots \cdot x_n \leq \frac{1}{2020^{2021}}$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{i=1}^{2021} \frac{1}{1 + x_i} = 2020 \Leftrightarrow \sum_{i=1}^{2021} \frac{x_i}{1 + x_i} = 1 \Rightarrow$$

$$\frac{1}{1 + x_i} = 1 - \frac{x_i}{1 + x_i} = \sum_{j \neq i} \frac{x_j}{1 + x_j} \overset{AM-GM}{\geq} 2020 \sqrt[2020]{\prod_{j \neq i} \frac{x_j}{1 + x_j}} \Rightarrow$$

$$\prod_{i=1}^{2021} \frac{1}{1 + x_i} \geq 2020^{2021} \sqrt[2021]{\prod_{i=1}^{2021} \prod_{j \neq i} \frac{x_j}{1 + x_j}} = 2020^{2021} \cdot (x_1 \cdot x_2 \cdot \dots \cdot x_n) \prod_{i=1}^{2021} \frac{1}{1 + x_i}$$

Therefore,

$$x_1 \cdot x_2 \cdot \dots \cdot x_n \leq \frac{1}{2020^{2021}}$$

Solution 2 by Marian Ursărescu-Romania

Because $\frac{1}{1+x_1} + \frac{1}{1+x_2} + \dots + \frac{1}{1+x_{2021}} = \frac{1}{2020} \Rightarrow \exists \alpha_1, \alpha_2, \dots, \alpha_{2021} \in \mathbb{R}$ **such that:**

$$x_1=\frac{\alpha_1}{\alpha_2+\alpha_3+\cdots+\alpha_{2021}},x_2=\frac{\alpha_2}{\alpha_1+\alpha_3+\cdots+\alpha_{2021}},\ldots,$$

$$x_{2021}=\frac{\alpha_{2021}}{\alpha_1+\alpha_2+\cdots+\alpha_{2020}}$$

We must show that:

$$\frac{\alpha_1\cdot\alpha_2\cdot\alpha_3\cdot\ldots\cdot\alpha_{2021}}{(\alpha_2+\alpha_3+\cdots+\alpha_{2021})(\alpha_1+\alpha_3+\cdots+\alpha_{2021})\cdot\ldots\cdot(\alpha_1+\alpha_2+\cdots+\alpha_{2020})}\le\frac{1}{2020^{2021}}$$

$$(\alpha_2+\alpha_3+\cdots+\alpha_{2021})(\alpha_1+\alpha_3+\cdots+\alpha_{2021})\cdot\ldots\cdot(\alpha_1+\alpha_2+\cdots+\alpha_{2020})\ge$$

$$\ge 2020^{2021}\cdot\alpha_1\cdot\alpha_2\cdot\alpha_3\cdot\ldots\cdot\alpha_{2021};(1)$$

$$\alpha_2+\alpha_3+\cdots+\alpha_{2021}\ge 2020\sqrt[2020]{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{2021}}$$

$$\begin{cases}\alpha_2+\alpha_3+\cdots+\alpha_{2021}\ge 2020\sqrt[2020]{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{2021}}\\ \alpha_1+\alpha_3+\cdots+\alpha_{2021}\ge 2020\sqrt[2020]{\alpha_1\alpha_3\cdot\ldots\cdot\alpha_{2021}}\\ \ldots\\ \alpha_1+\alpha_2+\cdots+\alpha_{2020}\ge 2020\sqrt[2020]{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{2020}}\end{cases}\Rightarrow (1)\text{ is true. Proved.}$$

234. $A,B,C\in M_n(\mathbb{R}),n\in\mathbb{N},n\ge 2,AB=BA,AC=CA,BC=CB.$

If $A+B+C=I_n$ **then** $det(AB+C)\cdot det(BC+A)\cdot det(CA+B)\ge 0$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$det(AB+C)=det(AB+I_n-A-B)=det(A(B-I_n)+I_n-B)=$$

$$=det\big((B-I_n)(A-I_n)\big)=det(A-I_n)det(B-I_n);(1)$$

$$det(BC+A)=det(BC+I_n-B-C)=det(B(C-I_n)+I_n-C)=$$

$$=det\big((C-I_n)(B-I_n)\big)=det(B-I_n)det(C-I_n);(2)$$

$$det(CA+B)=det(CA+I_n-A-C)=det(C(A-I_n)+I_n-A)=$$

$$=det\big((A-I_n)(C-I_n)\big)=det(A-I_n)det(C-I_n);(3)$$

From (1),(2),(3) it follows that:

$$det(AB+C)\cdot det(BC+A)\cdot det(CA+B)\ge\big(det(A-I_n)det(B-I_n)det(C-I_n)\big)^2\ge 0$$

Solution 2 by Alex Szoros-Romania

$$A+B+C=I_n\Rightarrow C=I_n-A-B$$

$$AB+C=AB+I_n-A-B=(A-I_n)(B-I_n)\Rightarrow$$
$$det(AB+C)=det(A-I_n)det(B-I_n);(1)$$
$$BC+A=BC+I_n-B-C=(B-I_n)(C-I_n)\Rightarrow$$
$$det(BC+A)=det(B-I_n)det(C-I_n);(2)$$
$$CA+B=CA+I_n-C-A=(C-I_n)(A-I_n)\Rightarrow$$
$$det(CA+B)=det(A-I_n)det(C-I_n);(3)$$

From (1),(2),(3) it follows that:

$$det(AB+C)\cdot det(BC+A)\cdot det(CA+B)\ge\left(det(A-I_n)det(B-I_n)det(C-I_n)\right)^2\ge 0$$

Solution 3 by Ravi Prakash-New Delhi-India

$$AB+C=AB+I_n-A-B=(A-I_n)(B-I_n)\Rightarrow$$
$$det(AB+C)=det(A-I_n)det(B-I_n);(1)$$
$$BC+A=BC+I_n-B-C=(B-I_n)(C-I_n)\Rightarrow$$
$$det(BC+A)=det(B-I_n)det(C-I_n);(2)$$
$$CA+B=CA+I_n-C-A=(C-I_n)(A-I_n)\Rightarrow$$
$$det(CA+B)=det(A-I_n)det(C-I_n);(3)$$

From (1),(2),(3) it follows that:

$$det(AB+C)\cdot det(BC+A)\cdot det(CA+B)\ge det^2(A-I_n)det^2(B-I_n)det^2(C-I_n)\ge 0$$

235. $A\in M_2(\mathbb{R}), TrA+detA=2$. **Prove that:**

$$det(A^2+detA\cdot A+TrA\cdot I_2)\ge 4$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$p(x)=x^2-TrAx+detA,\qquad \lambda_1+\lambda_2=TrA,\lambda_1\lambda_2=detA$$
$$det(A^2+detA\cdot A+trA\cdot I_2)=$$
$$=\left(\lambda_1^2+detA\lambda_1+TrA\right)\left(\lambda_2^2+detA\lambda_2+TrA\right)=$$
$$=(\lambda_1\lambda_2)^2+\lambda_1^2\lambda_2detA+\lambda_1^2TrA+detA\lambda_1\lambda_2^2+(detA)^2\lambda_1\lambda_2+detATrA\lambda_1+TrA\lambda_2^2$$
$$+TrAdetA\lambda_2+(TrA)^2=$$

$$= (detA)^2 + (detA)^2TrA + TrA((TrA)^2 - 2detA) + (detA)^3 + detA(TrA)^2 + (TrA)^2 =$$

$$= (TrA)^3 + (detA)^3 + (TrA)^2 + (detA)^2 + detATrA(detA + TrA) - 2detATrA =$$

$$= (TrA)^3 + (detA)^3 + (TrA)^2 + (detA)^2; (1)$$

$$(detA + TrA) = 2 \Rightarrow (TrA)^2 + (detA)^2 = 4 - 2TrAdetA; (2)$$

$$(detA + TrA) = 2 \Rightarrow (TrA)^3 + (detA)^3 = 8 - 6TrAdetA; (3)$$

From (1),(2),(3) it follows that:

$$det(A^2 + detA \cdot A + TrA \cdot I_2) = 12 - 8TrAdetA; (4)$$

$$4TrAdetA \leq (detA + TrA)^2 = 4 \Rightarrow TrAdetA \leq 1; (5)$$

From (4),(5) it follows that:

$$det(A^2 + detA \cdot A + TrA \cdot I_2) \geq 4$$

Solution 2 by Alex Szoros-Romania

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}), TrA = a + d \overset{not}{=} t; detA = ac - bd \overset{not}{=} \delta$$

$$A^2 - tA + \delta I_2 = O_2 \Rightarrow A^2 = tA - \delta I_2$$

$$A^2 + \delta A + tI_2 = tA - \delta I_2 + \delta A + tI_2 = 2A + (t - \delta)I_2 =$$

$$= 2A + (2 - 2\delta)I_2 = 2[A + (1 - \delta)I_2] = 4\begin{vmatrix} a + 1 - \delta & b \\ c & d + 1 - \delta \end{vmatrix}$$

$$det(A^2 + \delta A + tI_2) \geq 4 \Leftrightarrow (a + 1 - \delta)(d + 1 - \delta) - bc \geq 1 \Leftrightarrow$$

$$\delta^2 - \delta - t\delta + t \geq 0 \Leftrightarrow 2\delta^2 - 4\delta + 2 \geq 0 \Leftrightarrow (\delta - 1)^2 \geq 0$$

Solution 3 by Ravi Prakash-New Delhi-India

Let: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}), TrA = a + d; detA = ac - bd.$

We are given: $TrA + detA = 2$. Also, by the Cayley Hamilton theorem:

$$A^2 - TrA \cdot A + detA \cdot I_2 = O_2 \Rightarrow A^2 = TrA \cdot A - detA \cdot I_2$$

Now, $A^2 + detA \cdot A + TrA \cdot I_2 = TrA \cdot A - detA \cdot I_2 + detA \cdot A + TrA \cdot I_2 =$

$$= 2(A + (TrA - 1)I_2) = 2\begin{vmatrix} 2a + d & b \\ c & 2d + a - 1 \end{vmatrix} \Rightarrow$$

$$det(A^2 + detA \cdot A + TrA \cdot I_2) = 4(1 - 4(a + d) + 4ad + 2(a^2 + d^2) + 2) =$$

$$= 4(1 + 2(a + d - 1)^2) \geq 0$$

236. $z_1, z_2, z_3 \in \mathbb{C}^*$ **–different in pairs such that**

$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$. **If**

$$\sum_{cyc} \frac{1}{3|2z_1 - z_2 - z_3| + |2z_2 - z_1 - z_3|} = \frac{1}{4} \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution by proposer

$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3); \Delta ABC \subset C(0,1)$$

$$\sum_{cyc} \frac{1}{3|2z_1 - z_2 - z_3| + |2z_2 - z_1 - z_3|} = \frac{1}{4} \Leftrightarrow$$

$$\sum_{cyc} \frac{1}{2} \cdot \frac{1}{3\left|z_1 - \frac{z_2 + z_3}{2}\right| + \left|z_2 - \frac{z_1 + z_3}{2}\right|} = \frac{1}{4} \Leftrightarrow \sum_{cyc} \frac{1}{3m_a + m_b} = \frac{1}{2}; (1)$$

$$\frac{1}{2} = \sum_{cyc} \frac{1}{3m_a + m_b} \overset{CBS}{\geq} \frac{9}{4(m_a + m_b + m_c)}; (2)$$

But: $m_a + m_b + m_c \leq \frac{9R}{2} \overset{R=1}{=} \frac{9}{2}; (3)$

From (2),(3) we get: $\frac{1}{2} \geq \frac{9}{4(m_a+m_b+m_c)} \geq \frac{9}{4 \cdot \frac{9}{2}} = \frac{1}{2}$

$$m_a + m_b + m_c = \frac{9R}{2} \overset{R=1}{=} \frac{9}{2} \; (4)$$

From (3),(4) it follows that ΔABC –equilateral.

237. $z_1, z_2, z_3 \in \mathbb{C}^*$ **–different in pairs,** $|z_1| = |z_2| = |z_3|, A(z_1), B(z_2), C(z_3)$

$$\sum_{cyc} \left| \frac{(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2|}{2z_1 - z_2 - z_3} \right| = \sum_{cyc} |z_1 - z_2| \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let O –circumcenter of ΔABC is origin of the complex plane.

$|z_1| = |z_2| = |z_3| = R; |z_1 - z_2| = AB = c$ (and analogs)

$$(z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2| = |b(z_1 - z_2) + c(z_1 - z_3)| =$$

$$= |a+b+cz_1-(az_1+bz_2+cz_3)| =$$

$$= 2s\left|z_1-\frac{az_1+bz_2+cz_3}{a+b+c}\right| = 2s\cdot AI$$

$$|2z_1-z_2-z_3| = 2\left|z_1-\frac{z_2+z_3}{2}\right| = 2m_a$$

$$\sum_{cyc}\left|\frac{(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|}{2z_1-z_2-z_3}\right| = \sum_{cyc}|z_1-z_2| \Leftrightarrow \sum_{cyc}\frac{AI}{m_a} = 2$$

We know that:

$$\sum_{cyc}\frac{AI}{m_a} = r\sum_{cyc}\frac{1}{m_a\sin\frac{A}{2}} \le r\sum_{cyc}\frac{1}{w_a\sin\frac{A}{2}} = r\sum_{cyc}\frac{b+c}{2bc\cdot\cos\frac{A}{2}\sin\frac{A}{2}} =$$

$$= 2Rr\sum_{cyc}\frac{b+c}{abc} = 2$$

Therefore:

$$\sum_{cyc}\left|\frac{(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|}{2z_1-z_2-z_3}\right| = \sum_{cyc}|z_1-z_2| \Rightarrow AB = BC = CA$$

Equality holds if and only if triangle is equilateral.

Solution 2 by proposer

$$|z_1| = |z_2| = |z_3| = R, A(z_1), B(z_2), C(z_3), \Delta ABC \subset C(0,1)$$

$$\sum_{cyc}\left|\frac{(z_1-z_2)|z_1-z_3|+(z_1-z_3)|z_1-z_2|}{2z_1-z_2-z_3}\right| = \sum_{cyc}|z_1-z_2| \Leftrightarrow$$

$$\sum_{cyc}\left|\frac{(z_1-z_2)AC+(z_1-z_3)AB}{2\left(z_1-\frac{z_2+z_3}{2}\right)}\right| = AB+BC+CA \Leftrightarrow$$

$$\sum_{cyc}\frac{|(z_1-z_2)b+(z_1-z_3)c|}{\left|z_1-\frac{z_2+z_3}{2}\right|} = 2(a+b+c) \Leftrightarrow$$

$$\sum_{cyc}\frac{|(b+c)z_1-bz_2-cz_3|}{m_a} = 2(a+b+c) \Leftrightarrow$$

$$\sum_{cyc}\frac{|(a+b+c)z_1-(az_1+bz_2+cz_3)|}{m_a} = 2 \Leftrightarrow \sum_{cyc}\frac{AI}{m_2} = 2; (1)$$

But: $m_a \geq \sqrt{s(s-a)}$ and $AI = \frac{r}{\sin\frac{A}{2}} \Rightarrow$

$$\sum_{cyc}\frac{AI}{m_a} \leq \sum_{cyc}\frac{r}{\sqrt{s(s-a)}\sin\frac{A}{2}} = \sum_{cyc}\frac{r}{\sqrt{s(s-a)}\cdot\sqrt{\frac{(s-b)(s-c)}{bc}}} =$$

$$= \sum_{cyc}\frac{r\sqrt{bc}}{s} = \sum_{cyc}\frac{\sqrt{bc}}{s} \leq \sum_{cyc}\frac{b+c}{2s} = 2; (2)$$

From (1),(2) it follows that ΔABC –equilateral.

238. $\sqrt{(x+y)(4-z)} + \sqrt{(y+z)(2-t)} + \sqrt{(z+t)(4-x)} + \sqrt{(t+x)(2-y)} =$

$$= \sqrt{\frac{5s^2 - 24S + 144}{2}}; s = x+y+z+t$$

Find: $\Omega = \overline{xyzt}$

Proposed by George Florin Şerban-Romania

Solution 1 by proposer

$$\sqrt{(x+y)(4-z)} \overset{AM-GM}{\leq} \frac{x+y+4-z}{2}, \sqrt{(y+z)(2-t)} \overset{AM-GM}{\leq} \frac{y+z+2-t}{2},$$

$$\sqrt{(z+t)(4-x)} \overset{AM-GM}{\leq} \frac{z+t+4-x}{2}, \sqrt{(t+x)(2-y)} \overset{AM-GM}{\leq} \frac{t+x+2-y}{2}$$

Thus,

$$\sqrt{\frac{5s^2 - 24S + 144}{2}} \leq \frac{s+12}{2}$$

$$\sqrt{\frac{(2s)^2 + (12-s)^2}{2}} \leq \frac{2s + (12-s)}{2} \leq \sqrt{\frac{(2s)^2 + (12-s)^2}{2}}$$

Applying AM-GM inequality, equality holds when $2s = 12 - s, s = 4 \rightarrow$

$$x+y = 4-z, y+z = 2-t, z+t = 4-x, t+x = 2-y$$

$$x+y+z+t = 4, x+y+z = 4 \rightarrow t = 0, x+z+t = 4 \rightarrow y = 0, x = 2, \overline{xyzt} = 2020.$$

Solution 2 by Michael Sterghiou-Greece

$$\sqrt{(x+y)(4-z)}+\sqrt{(y+z)(2-t)}+\sqrt{(z+t)(4-x)}+\sqrt{(t+x)(2-y)}$$
$$=\sqrt{\frac{5s^2-24S+144}{2}};(1)$$

$x,y,z,t\in\mathbb{N}$. By AM-GM, $LHS_{(1)}\le\frac{s+12}{2}$ and $RHS_{(1)}\le\frac{s+12}{2}\to\frac{9}{4}(s-4)^2\le 0\to s=4$.

We see that $(x,y,z,t)=(2,0,2,0)$ is the only solution giving $LHS_{(1)}=8=RHS_{(1)}$ with $s=4$. Therefor, $\Omega=\overline{xyzt}=2020$.

239. Find $x\in(0,\pi)$ such that:

$$tan^{-1}(x+1)+tan^{-1}2+tan^{-1}\left(\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}\right)=x$$

Proposed by Daniel Sitaru-Romania

Solution by Mohammad Rostami-Kabul-Afghanistan

$$\begin{cases}tan^{-1}(x+1)=\alpha\\ tan^{-1}2=\beta\\ tan^{-1}\left(\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}\right)=\gamma\end{cases}\Rightarrow\begin{cases}tan\alpha=x+1\\ tan\beta=2\\ tan\gamma=\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}\end{cases}$$

$$\alpha+\beta+\gamma=x\Rightarrow tan(\alpha+\beta+\gamma)=tanx\Rightarrow$$

$$\frac{tan\alpha+tan\beta+tan\gamma-tan\alpha\cdot tan\beta\cdot tan\gamma}{1-tan\alpha\cdot tan\beta-tan\beta\cdot tan\gamma-tan\gamma\cdot tan\alpha}=tanx$$

$$\frac{(x+1)+2+\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}-2(x+1)\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}}{1-2(x+1)-(x+1)\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}-2\frac{x^2sinx-2xcosx-2sinx}{x^2cosx-2xsinx-2cosx}}=tanx$$

$$\frac{(x^3+7x^2-6)cosx+(-2x^3-3x^2-2x+2)sinx}{(-2x^3+x^2+10x+2)cosx+(-x^3+x^2+4x+6)sinx}=tanx\Rightarrow$$

$$\begin{cases}-2x^3-3x^2-2x+2=-2x^3+x^2+10x+2\\ x^3+7x-6=0\\ -x^3+x^2+4x+6=0\end{cases}$$

$$I)-2x^3-3x^2-2x+2=-2x^3+x^2+10x+2\Rightarrow 4x^2+12x=0$$

$$\Rightarrow\begin{cases}x=0\notin(0,\pi)\\ x=-3x\notin(0,\pi)\end{cases}$$

$$II)\ x^3+7x^2-6=0\Rightarrow (x^3+x^2)+(6x^2-6)=0\Rightarrow (x+1)(x^2+6x-6)=0$$

$$\Rightarrow\begin{cases} x+1=0\Rightarrow x=-1\notin(0,\pi)\\ x^2+6x-6=0\Rightarrow x=\dfrac{-6\pm2\sqrt{15}}{2}\Rightarrow\begin{cases}x=-3-\sqrt{15}\notin(0,\pi)\\ x=-3+\sqrt{15}\in(0,\pi)\end{cases}\end{cases}$$

$$III)-x^3+x^2+4x+6=0\Rightarrow -(x^3-27)+(x^2-9)+4(x-3)=0$$

$$(x-3)(-x^2-2x-2)=0\Rightarrow\begin{cases}x-3=0\Rightarrow x=3\in(0,\pi)\\ x^2+2x+2=0\overset{\Delta<0}{\Longrightarrow}\emptyset\end{cases}$$

240. **Let $\lambda\ge 2$ be positive real numbers. Solve for real numbers:**

$$\frac{(\lambda+1+x)(\lambda+x)(\lambda-1+x)}{(\lambda+1-x)(\lambda-x)(\lambda-1-x)}=\frac{3(9\lambda^2-1)}{1-\lambda^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(\lambda+1+x)(\lambda+x)(\lambda-1+x)}{(\lambda+1-x)(\lambda-x)(\lambda-1-x)}=\frac{3(9\lambda^2-1)}{1-\lambda^2};(*)$$

$$\Leftrightarrow\frac{(\lambda+x)((\lambda+x)^2-1)}{(\lambda-x)((\lambda-x)^2-1)}=\frac{3(9\lambda^2-1)}{1-\lambda^2}$$

$$\Leftrightarrow\frac{(\lambda+x)((\lambda+x)^2-1)}{(x-\lambda)(1-(\lambda-x)^2)}=\frac{3(9\lambda^2-1)}{1-\lambda^2}\Rightarrow x=2\lambda \text{ is a solution of } (*)$$

$$\Leftrightarrow\frac{x^3+3\lambda x^2+(3\lambda^2-1)x+\lambda^3-\lambda}{-x^3+3\lambda x^2-(3\lambda^2-1)x+\lambda^3-\lambda}=\frac{27\lambda^2-3}{1-\lambda^2}$$

$$\Leftrightarrow(26\lambda^2-2)x^3-(84\lambda^2-12\lambda)x^2+(78\lambda^4-32\lambda^2+2)x-28\lambda^5+32\lambda^3-4\lambda=0$$

$$\Leftrightarrow(x-2\lambda)[(13\lambda^2-1)x^2-(16\lambda^3-4\lambda)x+(7\lambda^4-8\lambda^2+1)]=0$$

$$\Leftrightarrow x=2\lambda \text{ or } (13\lambda^2-1)x^2-(16\lambda^3-4\lambda)x+(7\lambda^4-8\lambda^2+1)=0$$

$$\Delta=(16\lambda^3-4\lambda)^2-4(13\lambda^2-1)(7\lambda^4-8\lambda^2+1)=$$

$$=-29\lambda^6-79\lambda^4(\lambda^2-4)-67\lambda^2-(\lambda^2-4)<0;(\lambda\ge2)$$

Hence, $(13\lambda^2-1)x^2-(16\lambda^3-4\lambda)x+(7\lambda^4-8\lambda^2+1)=0$ does not admit a real solution, then $S=\{2\lambda\}$ is unique solution.

241. $x\in\left[0,\frac{\pi}{2}\right], n\in\{2,3,\dots\}, x^n+(e^x+x^3)^n\le\frac{1}{2^{\frac{n}{2}-1}}$

Find x and n.

Proposed by Pavlos Trifon-Greece

Solution by Chris Kyriazis-Greece

Let $f(x) = x^n + (e^x + x^3)^n, x \in \left[0, \frac{\pi}{2}\right], n \geq 2 \rightarrow$

$$f'(x) = nx^{n-1} + n(e^x + x^3)^{n-1}(e^x + 3x^2), \forall x \in \mathbb{R}$$

So, f –continuous at $\left[0, \frac{\pi}{2}\right]$, f –is strictly increasing. Then attains maximum at $x = 0$, so

$$f(x) \geq f(0) = 1, \forall x \in \left[0, \frac{\pi}{2}\right]$$

If there is one $x \in \left[0, \frac{\pi}{2}\right]$ such that $f(x) \leq \frac{1}{2^{\frac{n}{2}-1}}$ it holds then $\frac{1}{2^{\frac{n}{2}-1}} \geq 1 \rightarrow 2^{\frac{n}{2}-1} \leq 1 \rightarrow$

$\frac{n}{2} - 1 \leq 0 \rightarrow n \leq 2$ and how $n \geq 2 \rightarrow n = 2$, thus $f(x) \leq 1$.

But for every $x \in \left[0, \frac{\pi}{2}\right]$, we have $f(x) \geq 1$. So if there is a $x \in \left[0, \frac{\pi}{2}\right]$ such that $f(x) \leq 1$ it must be $f(x) = 1$. This holds if and only if $x = 0$ because f –strictly increasing.

So, $x = 0, n = 2$ easy to check that verify the conditions.

242. Solve for real numbers:

$$\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x + i^2)(x + j^2) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^3 x + 4\cot^5 x}{2\cot^3 x + 4\tan^5 x}\right) dx$$

Proposed by Daniel Sitaru-Romania

Solution by Adrian Popa-Romania

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^3 x + 4\cot^5 x}{2\cot^3 x + 4\tan^5 x}\right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\cot^3 x + 4\tan^5 x}{2\tan^3 x + 4\cot^5 x}\right) dx =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{\frac{3}{\tan^3 x} + 4\tan^5 x}{2\tan^3 x + \frac{4}{\tan^5 x}}\right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{\tan^2 x(2 + 4\tan^8 x)}{4 + 2\tan^8 x}\right) dx; (1)$$

$$So, I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^3 x + \frac{4}{\tan^5 x}}{\frac{2}{\tan^3 x} + 4\tan^5 x}\right) dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^8 x + 4}{\tan^2 x(2 + 4\tan^8 x)}\right) dx; (2)$$

$$(1) + (2): 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log 1\, dx = 0$$

$$\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x+i^2)(x+j^2)=$$

$$=\left(2020x+\frac{2020\cdot 2021\cdot 4041}{6}\right)\left(2021x+\frac{2021\cdot 2022\cdot 4043}{6}\right)=0$$

$$x_1=-\frac{2021\cdot 1347}{2}, x_2=-\frac{674\cdot 4043}{2}$$

Solution 2 by Serlea Kabay-Liberia

$$\omega(a,b)=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} log\left(\frac{2tan^a x+4cot^b x}{2cot^a x+4tan^b x}\right)dx \overset{x=\frac{\pi}{6}+\frac{\pi}{3}-u}{=}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} log\left(\frac{2cot^a u+4tan^b u}{2tan^a u+4cot^b u}\right)du \to 2\omega(a,b)=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} log1\,dx=0$$

$$\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x+i^2)(x+j^2)=\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x^2+xj^2+xi^2+j^2i^2)=$$

$$=\sum_{i=1}^{n}\left(2021x^2+2021xi^2+\frac{2021\cdot 2022\cdot 4043}{6}x^2+\frac{2021\cdot 2022\cdot 4043}{6}i^2\right)=$$

$$=2021\sum_{i=1}^{n}\left(x^2+xi^2+\frac{2022\cdot 4043}{6}x+\frac{2022\cdot 4043}{6}i^2\right)$$

$$40822420x^2+40822420x\frac{2022\cdot 4043+2021\cdot 4041}{6}$$

$$+\frac{2020\cdot 2021^2\cdot 2022\cdot 4043\cdot 4041}{36}=0$$

$$x_1=-1362491, x_2=-\frac{2722287}{2}$$

Solution 3 by Mohammad Rostami-Kabul-Afghanistan

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

$$\Omega=\sum_{i=1}^{2020}(x+i^2)\sum_{j=1}^{2021}(x+j^2)=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} log\left[\frac{2tan^3\left(\frac{\pi}{2}-x\right)+4cot^5\left(\frac{\pi}{2}-x\right)}{2cot^3\left(\frac{\pi}{2}-x\right)+4tan^5\left(\frac{\pi}{2}-x\right)}\right]dx=$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\cot^3 x + 4\tan^5 x}{2\tan^3 x + 4\cot^5 x}\right) dx$$

$$2\Omega 2\left(\sum_{i=1}^{2020} x + \sum_{i=1}^{2020} i^2\right)\left(\sum_{j=1}^{2021} x + \sum_{j=1}^{2021} j^2\right) =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left[\frac{2\tan^3 x + 4\cot^5 x}{2\cot^3 x + 4\tan^5 x} \cdot \frac{2\cot^3 x + 4\tan^5 x}{2\tan^3 x + 4\cot^5 x}\right]$$

$$\left(2020x + \frac{2020 \cdot 2021 \cdot 4041}{6}\right)\left(2021x + \frac{2021 \cdot 2022 \cdot 4043}{6}\right) = 0$$

$$x_1 = -\frac{2021 \cdot 1347}{2}, x_2 = -\frac{674 \cdot 4043}{2}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$Let\ I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^3 x + 4\cot^5 x}{2\cot^3 x + 4\tan^5 x}\right) dx;\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \to$$

$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{2\tan^3 x + 4\cot^5 x}{2\cot^3 x + 4\tan^5 x}\right) dx = -I \to 2I = 0 \to I = 0$$

$$\sum_{i=1}^{2020}\sum_{j=1}^{2021}(x+i^2)(x+j^2) = 0 \to \sum_{i=1}^{2020}(x+i^2)\sum_{j=1}^{2021}(x+j^2) = 0 \to$$

$$\sum_{i=1}^{2020}(x+i^2) = 0\ or\ \sum_{j=1}^{2021}(x+j^2) = 0$$

$$2020x + \frac{2020 \cdot 2021 \cdot 4041}{6} = 0\ or\ 2021x + \frac{2021 \cdot 2022 \cdot 4043}{6}$$

$$x_1 = -\frac{2021 \cdot 1347}{2}, x_2 = -\frac{674 \cdot 4043}{2}$$

243. Solve for real numbers:

$$2 \cdot \frac{1+x^2}{1-x^2} = \sqrt{1+x} + \sqrt{1-x}$$

Proposed by Marin Chirciu-Romania

Solution by Michael Sterghiou-Greece

$$2\cdot\frac{1+x^2}{1-x^2}=\sqrt{1+x}+\sqrt{1-x};(1)$$

Domain of (1) is $-1<x<1$**. By Jensen** $LHS_{(1)}\leq 2\sqrt{\frac{1+x+1-x}{2}}=2$**, so**

$\frac{1+x^2}{1-x^2}\leq 1\to\frac{2x^2}{1-x^2}\leq 0$ **which is only valid for** $x=0$**, only solution.**

244. Solve for real numbers:

$$\frac{(x+3)^2-20}{2(x+1)}=\sqrt{(x+1)(x-3)}$$

Proposed by George Florin Şerban-Romania

Solution 1 by Bedri Hajrizi-Mitrovica-Kosovo

$$(x+3)^2-20=2(x+1)\sqrt{(x+1)(x-3)}$$

$$x\in(-\infty,-1)\cup[3,\infty);(1)$$

$$(x+3)^2-20\geq 0,x\in\left(-\infty,-3-2\sqrt{5}\right]\cup\left[-3+2\sqrt{3},\infty\right);(2)$$

From $(1),(2)$: $x\in\left(-\infty,-3-2\sqrt{5}\right]\cup[3,\infty);(3)$

Let $x-1=t\to((t+4)^2-20)^2=4(t+2)^2(t+2)(t-2)$

$$(t^2+8t-4)^2=4(t^2+4t+4)(t^2-4)$$

$$t^4+16t^3+56t^2-64t+16=4t^4+16t^3-64t-64$$

$$3t^4-56t^2-80=0\to t^2=\frac{56\pm 64}{6}$$

$t^2=-\frac{4}{3}$**, no solution.**

$t^2=20\to t=\pm 2\sqrt{5}\to x=1\pm 2\sqrt{5}\to x_1=1-2\sqrt{5}$ **no solution.**

$x=1+2\sqrt{5}$ **solution.**

Solution 2 by proposer

$$(x+1)(x-3)\geq 0,x\in(-\infty,-1]\cup[3,\infty),x^2+6x-11=2(x+1)\sqrt{(x+1)(x-3)}$$

$$(x+1)^2+4(x-3)=2(x+1)\sqrt{(x+1)(x-3)}$$

$$(x+1)^2-2(x+1)\sqrt{(x+1)(x-3)}+4(x-3)=0$$

Let us denote: $x+1=t\to$

$$t^2-2t\sqrt{(x+1)(x-3)}+4(x-3)=0,\Delta=4(x-3)^2$$

If $t=\frac{2\sqrt{(x+1)(x-3)}+2(x-3)}{2}, x+1=\sqrt{(x+1)(x-3)}, x^2-2x-19=0$

$$(x-1)^2=20\to x-1=\sqrt{20}\to x_1\in[3,\infty), x-1=-\sqrt{20}\in(-\infty,-1]$$

If $t=\frac{2\sqrt{(x+1)(x-3)}-2(x-3)}{2}\to x+1=\sqrt{(x+1)(x-3)}-x+3\to$

$$2x-2=\sqrt{(x+1)(x-3)}\to 4x^2-8x+4=x^2-2x-3\to 3x^2-6x+7=0,$$

$\Delta=-48<0$. So, $S=\{\sqrt{20}+1\}$.

245. Let $f_m(x)=\frac{m}{sinx+2}, g_m(x)=\frac{m}{cosx+2}$. Find all real numbers m such that:

$$minf_m(x)+maxg_m(x)=1,\forall\in\left[0,\frac{\pi}{2}\right]$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Tran Hong-Dong Thap-Vietnam

$$f'_m(x)=-\frac{mcosx}{(sinx+2)^2}, g'_m(x)=\frac{msinx}{(cosx+2)^2}$$

If $m\geq 0,\forall m\in\left[0,\frac{\pi}{2}\right]$ then $f'_m(x)\leq 0\Rightarrow f_m\downarrow\left[0,\frac{\pi}{2}\right], g'_m(x)\geq 0\Rightarrow g_m\uparrow\left[0,\frac{\pi}{2}\right]\Rightarrow$

$$minf_m(x)=f_m\left(\frac{\pi}{2}\right)=\frac{m}{3}, maxg_m(x)=g_m\left(\frac{\pi}{2}\right)=\frac{m}{2}\Rightarrow$$

$\frac{m}{3}+\frac{m}{2}=1\Rightarrow\frac{5m}{6}=1\Rightarrow m=\frac{6}{5}$(true for with $m\geq 0$)

If $m<0,\forall x\in\left[0,\frac{\pi}{2}\right]$ then $f'_m(x)\geq 0\Rightarrow f_m\uparrow\left[0,\frac{\pi}{2}\right], g'_m(x)\leq 0\Rightarrow g_m(x)\downarrow\left[0,\frac{\pi}{2}\right]\Rightarrow$

$$minf_m(x)=f_m(0)=\frac{m}{2}, maxg_m(x)=g_m(0)=\frac{m}{3}\Rightarrow$$

$\frac{m}{3}+\frac{m}{2}=1\Rightarrow\frac{5m}{6}=1\Rightarrow m=\frac{6}{5}$(false for with $m<0$)

Therefore, $m=\frac{6}{5}$.

246. Solve for real numbers:

$$2\left(e^x - e^{\frac{1}{x}}\right) = \left(x - \frac{1}{x}\right)\left(e^x + x^{\frac{1}{x}}\right)$$

Proposed by Ionuţ Florin Voinea-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$2\left(e^x - e^{\frac{1}{x}}\right) = \left(x - \frac{1}{x}\right)\left(e^x + x^{\frac{1}{x}}\right) \Leftrightarrow \frac{e^x - e^{\frac{1}{x}}}{e^x + e^{\frac{1}{x}}} = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

$$\Leftrightarrow \frac{e^{x-\frac{1}{x}} - 1}{e^{x-\frac{1}{x}} + 1} = \frac{1}{2}\left(x - \frac{1}{x}\right) \Leftrightarrow tanh(y) = y; \because y = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

Let $f(y) = tanh(y) - y, y \in \mathbb{R}, f'(y) = -tanh^2 y \leq 0 \Rightarrow f$ –decreasing on $\mathbb{R} \Rightarrow$

$$f(y) = 0 \Leftrightarrow y = 0 \Leftrightarrow x - \frac{1}{x} = 0 \Leftrightarrow x = \pm 1$$

Therefore, $S = \{\pm 1\}$.

Solution 2 by Ruxandra Daniela Tonilă-Romania

$$2\left(e^x - e^{\frac{1}{x}}\right) = \left(x - \frac{1}{x}\right)\left(e^x + x^{\frac{1}{x}}\right) \Leftrightarrow$$

$$2e^{\frac{1}{x}}\left(e^{x-\frac{1}{x}} - 1\right) = \left(x - \frac{1}{x}\right)e^{\frac{1}{x}}\left(e^{x-\frac{1}{x}} + 1\right) \Leftrightarrow 2\left(e^{x-\frac{1}{x}} - 1\right) = \left(x - \frac{1}{x}\right)\left(e^{x-\frac{1}{x}} + 1\right)$$

Let $x - \frac{1}{x} = t \Rightarrow 2(e^t - 1) = t(e^t + 1)$

$$te^t + t + 2 - 2e^t = 0 \Leftrightarrow e^t(t - 2) + t + 2 = 0$$

Let $f: \mathbb{R} \to \mathbb{R}, f(t) = e^t(t - 2) + t + 2, f'(t) = e^t(t - 1) + 1$

$f'(t) = 0 \Leftrightarrow t = 0$ and f –strictly increasing thus, $t = 0$ is only solution.

Therefore,

$$x - \frac{1}{x} = 0 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1.$$

247. If $m, n \in \mathbb{N}$, L_n –Lucas numbers, F_m –Fibonacci numbers then:

$$\sqrt[3]{F_n F_m^2 L_m L_n^2} + \sqrt[3]{F_m F_n^2 L_n L_m^2} \leq 2F_{m+n}$$

Proposed by Daniel Sitaru-Romania

Solution by Noor Alam-India

$$\frac{\sqrt[3]{F_nF_m^2L_mL_n^2}+\sqrt[3]{F_mF_n^2L_nL_m^2}}{2}\le\sqrt[3]{\frac{F_nF_m^2L_mL_n^2+F_mF_n^2L_nL_m^2}{2}}$$

$$\sqrt[3]{\frac{F_nF_m^2L_mL_n^2+F_mF_n^2L_nL_m^2}{2}}=\sqrt[3]{\frac{F_nF_mL_nL_m(F_mL_n+F_nL_m)}{2}}=$$

$$=\sqrt[3]{\frac{F_nF_mL_nL_m\cdot 2F_{m+n}}{2}}=\sqrt[3]{F_nF_mL_nL_mF_{m+n}}$$

$$\because F_mL_n+F_nL_m=2F_{m+n}$$

Now, $F_n=\frac{\alpha^n-\beta^n}{\sqrt{5}}$, $L_n=\alpha^n+\beta^n$, where $\alpha=\frac{1+\sqrt{5}}{2}$, $\beta=\frac{1-\sqrt{5}}{2}$.

$$F_nF_mL_nL_m=\frac{\alpha^n-\beta^n}{\sqrt{5}}\cdot\frac{\alpha^m-\beta^m}{\sqrt{5}}\cdot(\alpha^n+\beta^n)\cdot(\alpha^m+\beta^m)=\frac{(\alpha^{2n}-\beta^{2n})(\alpha^{2m}-\beta^{2m})}{5}$$

$$=\frac{\alpha^{2(m+n)}-\alpha^{2m}\beta^{2n}+\alpha^{2n}\beta^{2m}+\beta^{2(m+n)}}{5}=\frac{(\alpha^{m+n}-\beta^{m+n})^2}{5}=$$

$$=\left(\frac{\alpha^{m+n}-\beta^{m+n}}{\sqrt{5}}\right)^2=F_{m+n}^2$$

Thus, $\sqrt[3]{\frac{F_nF_m^2L_mL_n^2+F_mF_n^2L_nL_m^2}{2}}=\sqrt[3]{F_{m+n}^3}=F_{m+n}$. Therefore,

$$\sqrt[3]{F_nF_m^2L_mL_n^2}+\sqrt[3]{F_mF_n^2L_nL_m^2}\le 2F_{m+n}$$

248. Find all functions $f:[-2020;2020]\to\mathbb{R}$ such that:

$$(f(x-y))^3=5f(x+2y)-x^3y^3,\forall x\in[-2020,2020]$$

Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$x=y=0\to(f(0))^3=5f(0)\to f(0)=0$ of $f(0)=\pm\sqrt{5}$. Let $a=f(0)$.

$$x=y\to a^3=5f(3x)-x^6\to f(3x)=\frac{1}{5}(x^6+a^3)\to$$

$$f(x)=\frac{1}{5}\left[\left(\frac{x}{3}\right)^6+a^3\right], \forall x\in[-2020,2020]$$

-with $a=0$ or $a=-\sqrt{5}$ or $x=\sqrt{5}$.

249. Find the remainder, when the number $2019^{2020^{2021}}$ is divided by 7.

Proposed by Rajeev Rastogi-India

Solution 1 by Mohamed Amine Ben Ajiba-Morocco

$$2019\equiv 3(mod\ 7)\to 2019^3\equiv -1(mod\ 7)\to 2019^2\equiv 1(mod\ 7)$$

$$2020\equiv 4(mod\ 6)\to 2019^{2020}\equiv 2019^4\equiv -3\equiv 4(mod\ 7)$$

Let $a=2019^{2020}$; $a\equiv 4(mod\ 7)\to a^3\equiv 1(mod\ 7)$

We know that $2020\equiv 1(mod\ 3)\to 2020^{2020}\equiv 1(mod\ 3)\to$

$$a^{2020^{2020}}\equiv a\equiv 4(mod\ 7)$$

$$\to 2019^{2020^{2021}}=a^{2020^{2020}}\equiv 4(mod\ 7)$$

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $7|(2019-3)\to 2019^{2020}\equiv 3^{2020}(mod\ 7)$ and since

$$3^1,3^2,3^33^4,3^5,3^6\equiv 3,2,6,4,5,1,(mod\ 7)$$

Hence, $3^{2020}\equiv 4^{2020}(mod\ 7), 2020\equiv 4(mod\ 6)\to (3^{2020})^{2020}\equiv 4^{2020}(mod\ 7)$ and since $4^1,4^2,4^3\equiv 4,3,1(mod\ 7)$ hence, $4^{2020}\equiv (mod\ 7), 2020\equiv 1\ (mod\ 3)\to$

$$2019^{2020^{2021}}=a^{2020^{2020}}\equiv 4(mod\ 7)$$

250. Given x be the least prime divisor of the number $1\underbrace{000\ldots0}_{2018-times}1$ also

$$(2x)^{(2x)^{(2x)}}\equiv k(mod\ 100)$$ then find k.

Proposed by Rajeev Rastogi-India

Solution by Surjeet Singhania-India

$$1\underbrace{000\ldots0}_{2018-times}1=10^{2019}+1\equiv(\mathrm{mod}\ 7)\to \mathrm{x}=7.$$

Denote $y=(2x)^{(2x)^{(2x)}}=14^{14^{14}}$. We need to find $y(mod\ 100)$.

Since $4|y$, now we need to find $y(mod25)$.

$y\equiv 14^{14^{14}}(mod\ 25)$. Since $\Phi(25)=20\to 14^{14}\equiv 6^{14}(mod\ 20)$

$14^{14} \equiv 16(mod\ 20) \to y \equiv 14^{16}(mod\ 25) \equiv 4^8(mod\ 25)(Euler\ th.)$. Hence,
$y \equiv 36(mod\ 25) \equiv 11(mod\ 25)$. Since $4|y \to y = 4k_1$.
$y = 4k_1 \equiv 11(mod\ 25) \to k_1 \equiv 9(mod 25) \to y = 100k_2 + 36$.
Hence, $y \equiv 14^{14^{14}} \equiv 36(mod\ 100) \to k = 36$.

251. Show that the last two digits of the followings

$$9^9, 9^{9^9}, 9^{9^{9^9}}, 9^{9^{9^{9^9}}}, 9^{9^{9^{9^{9^9}}}} \text{ is always } 89.$$

In general prove that the last two digits of $9 \uparrow\uparrow n = \underbrace{9^{9^{\cdot^{\cdot^{9}}}}}_{n\geq 2}$ is 89.

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Surjeet Singhania-India

Define a sequence $x_n = 9^{x_{n-1}}, \forall n \in N$ and $x_0 = 9$.
Claim: Every element of sequence have last two digits are 89.
Let's check for $n = 1, x_1 \equiv 1\ (mod 4)$ and $x_1 \equiv 4^3(mod 25) \equiv 14(mod 25)$.
Let's solve these congruence. Since $x_1 \equiv 1(mod\ 4) \to x_1 = 4k_1 + 1$ also
$x_1 \equiv 14(mod\ 25) \to 4k_1 \equiv 13(mod 25) \to k_1 \equiv 22(mod 25) \to k_1 = 25k_2 + 22$.
Put the value of k_1 in $x_1, x_1 = 100k_2 + 89 \to k_1 \equiv 89(mod 100)$.
On he hypothesis true for $n = 1$. Assume it is true for $n = k, x_k \equiv 89(mod 100) \to$
$x_k = 100m + 89, m \in Z$. Now, we shall prove statement for $n = k + 1$.
$x_{k+1} \equiv 9^{x_k} = 9^{100m+89}, x_{k+1} \equiv 1(mod 4)$. Now, we have to mod 25 for the number.
We know that $\Phi(25) = 20$ and $100m + 89 \equiv 9(mod 20) \to$
$x_{k+1} \equiv 9^9(mod 25)(Euler\ Theorem)$.
$x_{k+1} \equiv 9^9(mod\ 25) \equiv 14(mod 25)$. Since $x_{k+1} \equiv 1(mod 4)$.
For 6^{th} line of our solution $x_{k+1} \equiv 89(mod 100)$.
Conclusion. In general prove that the last two digits of $9 \uparrow\uparrow n = \underbrace{9^{9^{\cdot^{\cdot^{9}}}}}_{n\geq 2}$ is 89.

252. Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \sqrt{1-x} + \dfrac{y}{\sqrt{1-y}} = 2\sqrt{1+z} \\ \begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy \end{vmatrix} = 0 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$x,y,z>0 \text{ and } 1-x\geq 0, 1-y>0, 1+z\geq 0 \Rightarrow x\in(0,1], y\in(0,1), z\in(0,\infty)$$

$$\Delta=\begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy\end{vmatrix}=\begin{vmatrix} xy+yz+zx & xy+yz+zx & xy+yz+zx \\ xy & zx & yz \\ zx & yz & xy\end{vmatrix}=$$

$$=(xy+yz+zx)\begin{vmatrix} 1 & 1 & 1 \\ xy & xz & yz \\ xz & yz & xy\end{vmatrix}=(xy+yz+zx)\left(-x^2(y-z)^2-yz(z-x)(y-x)\right)=0$$

$$-x^2(y^2-2yz+z^2)-yz(zy-xz-xy+x^2)=0\Leftrightarrow$$

$$x^2yz+y^2xz+z^2xy=x^2y^2+x^2z^2+y^2z^2$$

But, we know that: $(2,2,0)>(2,1,1)\Rightarrow x^2y^2+x^2z^2+y^2z^2\geq x^2yz+y^2xz+z^2xy$

Equality holds when $x=y=z\Rightarrow \sqrt{1-x}+\frac{x}{\sqrt{1-x}}=2\sqrt{1+x}\Leftrightarrow$

$$\sqrt{1-x^2}=\frac{1}{2}\Leftrightarrow x=\frac{\sqrt{3}}{2}\Rightarrow x=y=z=\frac{\sqrt{3}}{2}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\Delta=\begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy\end{vmatrix}=\begin{vmatrix} xy+yz+zx & xy+yz+zx & xy+yz+zx \\ xy & zx & yz \\ zx & yz & xy\end{vmatrix}=$$

$$=(xy+yz+zx)\begin{vmatrix} 1 & 1 & 1 \\ xy & xz & yz \\ xz & yz & xy\end{vmatrix}=0\Leftrightarrow\sum_{cyc}x^2(y-z)^2=0\Leftrightarrow x=y=z;(x,y,z>0)$$

$$\sqrt{1-x}+\frac{x}{\sqrt{1-x}}=2\sqrt{1+x}\Leftrightarrow\frac{x^2}{1-x}=3x+3$$

$$\sqrt{1-x^2}=\frac{1}{2}\Leftrightarrow x=\frac{\sqrt{3}}{2}\Rightarrow xy=z=\frac{\sqrt{3}}{2}$$

Solution 3 by Remus Florin Stanca-Romania

$$\begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy\end{vmatrix}=0\Leftrightarrow 3(xy)(yz)(zx)=(xy)^3+(xz)^3+(yz)^3, x,y,z>0$$

$$\Rightarrow xy=yz=zx\Rightarrow x=y=z\Rightarrow\sqrt{1-x}+\frac{x}{\sqrt{1-x}}=2\sqrt{1+x}\Leftrightarrow$$

$$4x^2=3, x>0\Leftrightarrow x=\frac{\sqrt{3}}{2}\Rightarrow xy=z=\frac{\sqrt{3}}{2}$$

Solution 4 by Ravi Prakash-New Delhi-India

$$\Delta=\begin{vmatrix} yz & xy & zx \\ xy & zx & yz \\ zx & yz & xy \end{vmatrix} = (xy+yz+zx)\begin{vmatrix} 1 & xy & zx \\ 1 & zx & yz \\ 1 & yz & xy \end{vmatrix} =$$

$$= (xy+yz+zx)\begin{vmatrix} 1 & xy & zx \\ 0 & x(z-y) & z(y-x) \\ 0 & z(y-x) & y(x-y) \end{vmatrix} = 0$$

$$\Rightarrow (xy+yz+zx)[xy(z-y)(x-z)-z^2(y-x)^2] = 0; (xy+yz+zx>0)$$

$$x^2yz + xyz^2 + xy^2z - x^2y^2 - z^2x^2 - z^2y^2 = 0 \Leftrightarrow$$

$$(xy-yz)^2 + (yz-zx)^2 + (zx-xy)^2 = 0 \Leftrightarrow xy = yz = zx \Leftrightarrow x = y = z$$

Equality holds when $x = y = z \Rightarrow \sqrt{1-x} + \frac{x}{\sqrt{1-x}} = 2\sqrt{1+x} \Leftrightarrow$

$$\sqrt{1-x^2} = \frac{1}{2} \Leftrightarrow x = \frac{\sqrt{3}}{2} \Rightarrow x = y = z = \frac{\sqrt{3}}{2}$$

253. Solve for real numbers:

$$\begin{cases} \left(\dfrac{x^3}{y} + xy + \dfrac{y^3}{x}\right)^2 = \sqrt{27(x^8+y^8+x^4y^4)} \\ \dfrac{5x^4-10xy+1}{x^4y^2-10y^4+5x^2} = \dfrac{y}{x^2} \end{cases}$$

Proposed by Orlando Irahola Ortega-Tarija-Bolivia

Solution 1 by Carlos Eduardo Aguiar Paiva-Fortaleza-Brazil

$$\left(\frac{x^3}{y} + xy + \frac{y^3}{x}\right)^2 = \sqrt{27(x^8+y^8+x^4y^4)}, (i)$$

$$\frac{5x^4-10xy+1}{x^4y^2-10y^4+5x^2} = \frac{y}{x^2}, (ii)$$

Now, $\left(\frac{x^2}{y^2} + \frac{y^2}{x^2} + 1\right)^2 = \sqrt{27\left(\frac{x^4}{y^4} + \frac{y^4}{x^4} + 1\right)}$

Let $m = \frac{x^2}{y^2} + \frac{y^2}{x^2} \Rightarrow (m+1)^2 = \sqrt{27(m^2-1)}$

$$\sqrt{m+1}\left(\sqrt{(m+1)^3} - 3\sqrt{3(m-1)}\right) = 0 \Rightarrow \sqrt{m+1} = 0 \Rightarrow m_1 = -1$$

$$\sqrt{(m+1)^3} = 3\sqrt{3(m-1)} \Rightarrow (m+1)^3 = 27(m-1) \Rightarrow (m-2)^2(m+7) = 0 \Rightarrow$$

$$m_{2,3}=2,m_4=-7$$

$$1)\ \frac{x^2}{y^2}+\frac{y^2}{x^2}=-1\Rightarrow\left(\frac{x}{y}+\frac{y}{x}\right)^2=1\Rightarrow\frac{x}{y}+\frac{y}{x}=\pm1\le2,\text{but }\ \frac{x}{y}+\frac{y}{x}\overset{AM-GM}{\ge}2$$

$$2)\ \frac{x^2}{y^2}+\frac{y^2}{x^2}=-7\Rightarrow\left(\frac{x}{y}+\frac{y}{x}\right)^2=-5\Rightarrow\frac{x}{y}+\frac{y}{x}=\pm5i\in\mathbb{C}$$

$$3)\ \frac{x^2}{y^2}+\frac{y^2}{x^2}=2\Rightarrow\left(\frac{x}{y}+\frac{y}{x}\right)^2=4\Rightarrow\frac{x}{y}+\frac{y}{x}=\pm2\Rightarrow y=x\text{ or }y=-x$$

$$ii)\ \frac{5x^4-10x^2+1}{x^6-10x^4+5x^2}=\frac{x}{x^2},x\ne0\Rightarrow x^5-5x^4-10x^3+10x^2+5x-1=0$$

$$(x-1)(x^4-4x^3-14x^2-4x+1)=0\Rightarrow x_1=1$$

$$x^4-4x^3-14x^2-4x+1=0\Rightarrow x^4-4x^3+4x^2=18x^2+4x-1$$

$$(x^2-2x+\lambda)^2=(2\lambda+18)x^2-(4\lambda-4)x+\lambda^2-1\Rightarrow\Delta_{RHS}=0$$

$$[-(4\lambda-4)]^2-4(2\lambda+18)(\lambda^2-1)=0\Rightarrow(\lambda-1)(\lambda^2+8\lambda+11)=0$$

$$\Rightarrow\lambda_1=1\Rightarrow(x^2-2x+1)^2=20x^2\Rightarrow x^2-2x+1=\pm2x\sqrt5$$

$$x_{2,3}=1+\sqrt5\pm\sqrt{5+2\sqrt5},x_{4,5}=1-\sqrt5\pm\sqrt{5-2\sqrt5}$$

$$ii)\ \frac{5x^4-10x^2+1}{x^6-10x^4+5x^2}=\frac{x}{x^2},x\ne0\Rightarrow x^5+5x^4-10x^3+10x^2+5x+1=0$$

By Newton-Raphson Method, $x_6=-6.699\dots$

$$(x,y)\in\left\{(1,1),\left(1+\sqrt5\pm\sqrt{5+2\sqrt5},1+\sqrt5\pm\sqrt{5+2\sqrt5}\right),\left(1-\sqrt5\pm\sqrt{5-2\sqrt5},1-\sqrt5\pm\sqrt{5-2\sqrt5}\right)\right\}$$

Solution 2 by Mohamed Amine Ben Ajiba-Morocco

$$\left(\frac{x^3}{y}+xy+\frac{y^3}{x}\right)^2=\sqrt{27(x^8+y^8+x^4y^4)},(1)$$

$$\frac{5x^4-10xy+1}{x^4y^2-10y^4+5x^2}=\frac{y}{x^2},(2)$$

$$(1)\Leftrightarrow\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}+1\right)^4=27\left(\frac{x^4}{y^4}+\frac{y^4}{x^4}+1\right)$$

Let us prove that: $(\sum a)^6 \geq 27(\sum ab)^2(\sum a^2), \forall a, b, c > 0, (*)$

Denote: $(p, q, r) = (\sum a, \sum ab, abc)$

$(*) \Leftrightarrow p^6 \geq 27q^2(p^2 - 2q) \Leftrightarrow (p^2 - 3q)^2(p^2 + 6q) \geq 0$ is true.

Equality holds if $p^2 = 3q \Leftrightarrow a = b = c$.

$$a = \frac{x^2}{y^2}, b = \frac{y^2}{x^2}, c = 1 \Rightarrow \sum ab = \frac{x^2}{y^2} + \frac{y^2}{x^2} + 1 = \sum a$$

$$(*) \Rightarrow \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} + 1\right)^4 \geq 27\left(\frac{x^4}{y^4} + \frac{y^4}{x^4} + 1\right)$$

Equality holds if $\frac{x^2}{y^2} = \frac{y^2}{x^2} = 1 \Rightarrow y = x$ or $y = -x$.

If $x = y \overset{(2)}{\Rightarrow} \frac{5x^4-10x^2+1}{x^5-10x^3+5x} = 1$, let us denote $x = tana \Rightarrow tan(5a) = \frac{tan^5a-10tan^3a+5tana}{5tan^4a-10tan^2a+1} = 1$

$$\Rightarrow a \in \left\{\frac{\pi}{20} + \frac{k\pi}{5} \middle| k \in \mathbb{Z}\right\} \Rightarrow$$

$$x = y = tan\left(\frac{\pi}{20}\right) = 1 + \sqrt{5} + \sqrt{5 + 2\sqrt{5}}$$

$$x = y = tan\left(\frac{\pi}{4}\right) = 1$$

$$x = y = tan\left(\frac{9\pi}{20}\right) = 1 + \sqrt{5} - \sqrt{5 + 2\sqrt{5}}$$

$$x = y = tan\left(\frac{13\pi}{20}\right) = 1 - \sqrt{5} - \sqrt{5 - 2\sqrt{5}}$$

$$x = y = tan\left(\frac{17\pi}{20}\right) = 1 - \sqrt{5} + \sqrt{5 - 2\sqrt{5}}$$

If $y = -x \overset{(2)}{\Rightarrow} \frac{5x^4+10x^2+1}{x^5-10x^3+5x} = -1 \Leftrightarrow x^5 + 5x^4 - 10x^3 + 10x^2 + 5x + 1 = 0$

Let $f(x) = x^5 + 5x^4 - 10x^3 + 10x^2 + 5x + 1 \Rightarrow$

$$f'(x) = 5\left(x^2 + \left(2\sqrt{3} + 2\right)x + 1\right)\left(x^2 - \left(2\sqrt{3} - 2\right)x + 1\right)$$

$$f'(x) = 0 \Leftrightarrow x_1 = -\sqrt{3} - 1 - \sqrt{3 + 2\sqrt{3}} \cong -5,27$$

$$x_2 == -\sqrt{3} - 1 + \sqrt{3 + 2\sqrt{3}} \cong -0,18$$

We have: $f(x_1) > 0, f(x_2) > 0 \Rightarrow f(x) = 0$ admits only one solution $x_0 < x_1$

By Newton-Raphson Method, $x_0 = -y_0 = -6.699\ldots$

$$(x,y)\in\left\{(1,1),\left(1+\sqrt{5}\pm\sqrt{5+2\sqrt{5}},1+\sqrt{5}\pm\sqrt{5+2\sqrt{5}}\right),\left(1-\sqrt{5}\pm\sqrt{5-2\sqrt{5}},1-\sqrt{5}\pm\sqrt{5-2\sqrt{5}}\right),(-6,699,6,699)\right\}$$

254. Solve in $\mathbb{C}$:

$$\begin{cases}(x+y)^2=5+xy\\ 9x^3-5x+2xy^2=26y^3+5y-2x^2y\end{cases}$$

Proposed by Carlos Paiva-Fortalezza-Brazil

Solution by Amir Sofi-Pristina-Kosovo

$$9x^3-5x+2xy^2=26y^3+5y-2x^2y\leftrightarrow$$

$$x^3+y^3-5(x+y)+2xy(x+y)+(2x)^3-(3y)^3=0\leftrightarrow$$

$$(+y)(x^2-xy+y^2+2xy-5)+(2x)^3-(3y)^3=0\leftrightarrow$$

$$(x+y)[(x+y)^3-xy-5]+(2x)^3-(3y)^3=0\leftrightarrow$$

$$(2x)^3-(3y)^3=0\leftrightarrow(2x-3y)(4x^2+6xy+9y^2)=0$$

$$4x^2+6xy+9y^2=0\leftrightarrow\frac{2x}{3y}+\frac{3y}{2x}+1=0\leftrightarrow\left(\frac{3y}{2x}\right)^2+\frac{3y}{2x}+1=0\leftrightarrow$$

$$\frac{3x}{2y}=\frac{-1\pm i\sqrt{3}}{2}\leftrightarrow y=\frac{-1\pm i\sqrt{3}}{3}x$$

$$(x+y)^2=5+xy$$

$$\left(x+\frac{-x+i\sqrt{3}x}{3}\right)^2=5+x\cdot\frac{-x+i\sqrt{3}x}{3}\ \text{v}\ \left(x+\frac{-x-i\sqrt{3}x}{3}\right)^2=5+x\cdot\frac{-x-i\sqrt{3}x}{3}$$

$$\left(\frac{2x+i\sqrt{3}x}{3}\right)^2=\frac{15-x^2+i\sqrt{3}x^2}{3}\ \text{v}\ \left(\frac{2x-i\sqrt{3}x}{3}\right)^2=\frac{15-x^2-i\sqrt{3}x^2}{3}$$

$$4x^2+i\sqrt{3}x^2-45=0\ \text{v}\ 4x^2-i\sqrt{3}x^2-45=0$$

$$x=\pm\sqrt{\frac{45(4-i\sqrt{3})}{19}}\ \text{v}\ x=\pm\sqrt{\frac{45(4+i\sqrt{3})}{19}}$$

$$(x,y)=\left(\pm\sqrt{\frac{45(4-i\sqrt{3})}{19}},\pm(-1\pm i\sqrt{3})\pm\sqrt{\frac{45(4-i\sqrt{3})}{19}}\right),$$

$$\left(\pm\sqrt{\frac{45(4+i\sqrt{3})}{19}},\pm(-1\pm i\sqrt{3})\pm\sqrt{\frac{45(4+i\sqrt{3})}{19}}\right)$$

255. Solve for real numbers:

$$\begin{cases} x,y,z>0 \\ \sqrt{xy}+\sqrt{yz}+\sqrt{zx}+\sqrt{\frac{x^2+y^2}{2}}+\sqrt{\frac{y^2+z^2}{2}}+\sqrt{\frac{z^2+x^2}{2}}=6 \\ x+y+z=3 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}+\sqrt{\frac{x^2+y^2}{2}}+\sqrt{\frac{y^2+z^2}{2}}+\sqrt{\frac{z^2+x^2}{2}}=6\Leftrightarrow$$

$$\sum_{cyc}\sqrt{\frac{x^2+y^2}{2}}-\sum_{cyc}x=\sum_{cyc}x-\sum_{cyc}\sqrt{xy};(1)$$

$$\sqrt{\frac{x^2+y^2}{2}}\ge\frac{x+y}{2}\Rightarrow\sum_{cyc}\sqrt{\frac{x^2+y^2}{2}}\ge\sum_{cyc}\frac{x+y}{2}=\frac{\sum x+\sum y}{2}=\sum_{cyc}x$$

$$\sqrt{xy}\le\frac{x+y}{2}\Rightarrow\sum_{cyc}\sqrt{xy}\le\sum_{cyc}x$$

So, $LHS\ge 0, RHS\ge 0$, equality holds when $x=y=z$, but $\sum x=3$, thus $x=y=z=1$.

Solution 2 by Florentin Vişescu-Romania

$$m_g+m_q\le 2m_a\text{, equality for } a=b.$$

$$\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\le a+b\Leftrightarrow ab+\frac{a^2+b^2}{2}+2\sqrt{\frac{a^3b+ab^3}{2}}\le a^2+b^2+2ab\Leftrightarrow$$

$$2\sqrt{\frac{a^3b+ab^3}{2}}\le\frac{a^2+b^2+2ab}{2}\Leftrightarrow\frac{4(a^3b+ab^3)}{2}\le\frac{(a+b)^4}{2}\Leftrightarrow$$

$(a-b)^4\ge 0$, with equality for $a=b$.

$$\begin{cases}\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}\le x+y\\ \sqrt{yz}+\sqrt{\frac{y^2+z^2}{2}}\le y+z\\ \sqrt{zx}+\sqrt{\frac{z^2+x^2}{2}}\le z+x\end{cases}\Rightarrow$$

$$\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}+\sqrt{yz}+\sqrt{\frac{y^2+z^2}{2}}+\sqrt{zx}+\sqrt{\frac{z^2+x^2}{2}}\le 2(x+y+z)$$

Equality holds for $x=y=z$, but $x+y+z=3\Rightarrow x=y=z=1$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}x=3=\frac{1}{2}\sum_{cyc}\left(\sqrt{xy}+\sqrt{\frac{x^2+y^2}{2}}\right)\overset{C-S}{\le}\frac{1}{2}\sum_{cyc}\sqrt{2\left(xy+\frac{x^2+y^2}{2}\right)}=\frac{1}{2}\sum_{cyc}(x+y)$$

$$=\sum_{cyc}x\Rightarrow xy=\frac{x^2+y^2}{2}(and\ anaogs)$$

Equality if and only if $x=y=z$, and how $\sum_{cyc}x=3$, thus $x=y=z=1$.

256. Solve for natural numbers:

$$x^{2019}+3\cdot y!=2161\cdot 2020^{3z}$$

Proposed by George Florin Şerban-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$x^{2019}+3\cdot y!=2161\cdot 2020^{3z};(*)$$

For $y\ge 7; 3y!\equiv 0\ (mod\ 7)$

$$2020\equiv 4\ (mod\ 7)\Rightarrow 2020^3\equiv 1(mod\ 7)\Rightarrow 2020^{3z}\equiv 1(mod\ 7)$$

$$2161\equiv 5\ (mod\ 7)\Rightarrow 2161\cdot 2020^{3z}\equiv 5(mod\ 7)$$

$$x^{2019} = (x^{673})^3 = a^3 \equiv 0;1;6\ (mod\ 7)$$

$$\Rightarrow \forall y \geq 7, x^{2019} + 3y! \neq 2161 \cdot 2020^{3z}$$

If $y = 0$ or $y = 1$ then, $(*) \Leftrightarrow x^{2019} = 2161 \cdot 2020^{3z} - 3.$

But $2161 \cdot 2020^{3z} - 3 \equiv 2\ (mod\ 7)$ and $x^{2019} \equiv 0;1;6\ (mod\ 7)$.

If $z = 0, (*) \Leftrightarrow^{2019} + 3y! = 2161 < 2^{12} \Rightarrow x < 2$

$$x = 0 \Rightarrow 3y! = 2161$$

$$x = 1 \Rightarrow 3y! = 2160 \Rightarrow y = 6.$$

If $z \neq 0; y \in \{2,3,4,5,6\}, x^{2019} = 2161 \cdot 2020^{3z} - 3y! \Rightarrow \forall x \in \mathbb{N}, x$ –eveb, then

$$\{x = 2t | t \in \mathbb{N}\} \Rightarrow 3y! = 2^6\left(2161 \cdot 2^{6(z-1)} \cdot 505^{3z} - 2^{2013} \cdot t^{2019}\right)$$

$\Rightarrow 2^6 | y!$ absurd for $y \leq 6$.

Therefore, $S = \{(1;6;0) | x, y, z \in \mathbb{N}\}$.

257. $k, n \in \mathbb{N}^*$ –fixed. Solve for real numbers:

$$x^{2k} - 2x^k + 3 = \sqrt[2n]{x} + \sqrt[2n]{2-x}$$

Proposed by Marin Chirciu-Romania

Solution by George Florin Şerban-Romania

$$\begin{cases} x \geq 0 \\ 2 - x \geq 0 \\ x^{2k} - 2x^k + 3 \geq 0 \end{cases} \Rightarrow \begin{cases} x \geq 0 \\ x \leq 2 \\ (x^k - 1)^2 + 2 \geq 0 \end{cases} \Rightarrow x \in [0,2]$$

Let $f(x) = \sqrt[2n]{x}, f: [0,2] \to \mathbb{R}, f'(x) = \frac{1}{2n} x^{\frac{1}{2n}-1}, f''(x) = \frac{1-2n}{4n^2} x^{\frac{1}{2n}-1} < 0, \forall x \in [0,2]$,

$$n \geq 1 \Rightarrow f \text{ –concave} \Rightarrow f\left(\frac{a+b}{2}\right) \geq \frac{f(a)+f(b)}{2} \Rightarrow f\left(\frac{x+2-x}{2}\right) \geq \frac{f(x)+f(2-x)}{2}$$

$$f(x) + f(2-x) = \sqrt[2n]{x} + \sqrt[2n]{2-x} \leq 2f(1) = 2$$

$$(x^k - 2) + 2 \leq 2 \Leftrightarrow 0 \leq (x^k - 1)^2 \leq 0 \Rightarrow x^k - 1 = 0 \Leftrightarrow x = 1.$$

258. Solve for real numbers:

$$2 \cdot \sqrt[4]{e^{4^x} \cdot e^{2 \cdot 6^x} \cdot e^{9^x}} = e^{4^x} + e^{9^x}$$

Proposed by Daniel Sitaru-Romania

Solution by Michael Sterghiou-Greece

$$2 \cdot \sqrt[4]{e^{4^x} \cdot e^{2 \cdot 6^x} \cdot e^{9^x}} = e^{4^x} + e^{9^x}; (1)$$

$$e^{4^x}\cdot e^{2\cdot 6^x}\cdot e^{9^x}=e^{4^x+2\cdot 6^x+9^x}=e^{(2^x)^2+(3^x)^2+2\cdot 2^x\cdot 3^x}=2^{(2^x+3^x)^2}\Rightarrow$$

$$LHS_{(1)}=2\cdot e^{\left(\frac{2^x+3^x}{2}\right)^2};(2)$$

By AM-GM inequality, we have:

$$RHS_{(1)}\geq 2\cdot\sqrt{e^{2^{2x}+3^{2x}}}=2\cdot e^{\frac{2^{2x}+3^{2x}}{2}}$$, therefore, we get:

$$e^{\left(\frac{2^x+3^x}{2}\right)^2}\geq e^{\frac{2^{2x}+3^{2x}}{2}}$$ and as $f(t)=e^t\uparrow$in $\mathbb{R}\Rightarrow\left(\frac{2^x+3^x}{2}\right)^2\geq\frac{2^{2x}+3^{2x}}{2}\Rightarrow$

$$\frac{2^{2x}+3^{2x}+2\cdot 6^x}{4}\geq\frac{2^{2x}+3^{2x}}{2}\Rightarrow 2^{2x}+3^{2x}-2\cdot 6^x\leq 0\Rightarrow(2^x-3^x)^2\leq 0$$

$\Leftrightarrow 2^x=3^x\Leftrightarrow x=0$ which is only solution.

259.

$$f_m(x)=(m+1)x^3-2(m+1)x^2-(m-2)x+2m-3, m\in\mathbb{R}-\{-1\}$$

Find the equation of the line which contains the three fixed points of f_m (the points not depends of m)

Proposed by Costel Florea-Romania

Solution 1 by Adrian Popa-Romania

$$f_m(x)=(m+1)x^3-2(m+1)x^2-(m-2)x+2m-3, m\in\mathbb{R}-\{-1\}$$

$$f_m(x)=mx^3+x^3-2mx^2-2x^2-mx+2m-3$$

$$f_m(x)=m(x^3-2x^2-x+2)+x^3-2x^2+2x-3$$

$$x^3-2x^2-x+2=0\Rightarrow x_1=1, x_2=2, x_3=-1$$

$$\Rightarrow A(1,-2), B(2,1), C(-1,-8)$$

$$AB:\frac{x-1}{2-1}=\frac{y+2}{1+2}\Rightarrow y=3x-5\Rightarrow C\in AB$$

Solution 2 by Alex Szoros-Romania

$$f_m(x)=(m+1)x^3-2(m+1)x^2-(m-2)x+2m-3, m\in\mathbb{R}-\{-1\}$$

$$f_m(x)=mx^3+x^3-2mx^2-2x^2-mx+2m-3$$

$$f_m(x)=m(x^3-2x^2-x+2)+x^3-2x^2+2x-3$$

$$f_m(x)=(m+1)(x^3-2x^2-x+2)+3x-5$$

If $P(x_0,y_0)$ –fix point if and only if $x_0^3-2x_0^2-x_0+2=0$, so $y=3x-5$.

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$f_m(x)=(m+1)x^3-2(m+1)x^2-(m-2)x+2m-3, m\in\mathbb{R}-\{-1\}$$

$$y=mx^3+x^3-2mx^2-2x^2-mx+2m-3$$

$$m(x^3-2x^2-x+2)+x^3-2x^2+2x-3-y=0$$

$$\begin{cases} x^3-2x^2-x+2=0, (1) \\ x^3-2x^2+2x-3-y=0, (2) \end{cases} \Rightarrow \begin{cases} x_1=1, x_2=2, x_3=-1 \\ A(1,-2), B(2,1), C(-1,-8) \end{cases}$$

$$AB: y=mx+p, m=\frac{y_B-y_A}{x_B-x_A}=3$$

$$A(2,1)\in(AB): 1=6+p\Rightarrow p=-5.$$

$$AB: y=3x-5$$

260. Find $x\in\mathbb{Z}$ such that: $\sqrt{(x-3)(5x-1)(5x^2-12x+3)}\in\mathbb{N}$.

Proposed by George Florin Şerban-Romania

Solution by Mohamed Amine Ben Ajiba-Morocco

$$(x-3)(5x-1)(5x^2-12x+3)=(5x^2-16x+3)(5x^2-12x+3)=$$

$$=\left((5x^2-14x+3)-2x\right)\left((5x^2-14x+3)+2x\right)=$$

$$=(5x^2-14x+3)^2-4x^2<(5x^2-14x+3)^2, \forall x\in\mathbb{Z}^*$$

$$(5x^2-14x+3)^2-4x^2>(5x^2-14x+2)^2$$

$$\Leftrightarrow (5x^2-14x+2)^2+2(5x^2-14x+2)+1-4x^2>(5x^2-14x+2)^2$$

$$\Leftrightarrow 6x^2-28x+5>0 \text{ true for all } x\in\mathbb{Z}-[1;4]$$

$$(5x^2-14x+2)^2<(x-3)(5x-1)(5x^2-14x+3)^2$$

$$\Rightarrow \forall x\in\mathbb{Z}-\{1,2,3,4\}, \sqrt{(x-3)(5x-1)(5x^2-12x+3)}\in\mathbb{N}.$$

$$x=0;\ \sqrt{9}=3\in\mathbb{N}, x=1; \sqrt{32}\notin\mathbb{N}$$

$$x=2;\ \sqrt{9}=3\in\mathbb{N}$$

$$x=3; 0\in\mathbb{N}$$

$$x=4;\ \sqrt{665}\notin\mathbb{N}\Rightarrow S=\{0,1,2,3,\}$$

261. Two numbers a and b are chosen uniformly and randomly from $(0,1)$. Then $\frac{a}{b}$ is calculated and rounded off to the nearest integer. Find the probability that after rounding off, the integer is odd (find the closed form).

Proposed by Arghyadeep Chatterjee-India

Solution by proposer

Probability $= P$

$$P=\sum_{r=1}^{\infty}\int_0^1\int_{\frac{2y}{4r+3}}^{\frac{2y}{4r+1}}dxdy+\int_0^1\int_{\frac{x}{2}}^{x}dydx+\int_0^1\int_{\frac{2}{3}y}^{y}dxdy=$$

$$=\frac{1}{4}+\frac{1}{6}+\sum_{r=1}^{\infty}\left(\frac{1}{4r+1}-\frac{1}{4r+3}\right)=\frac{1}{4}+\frac{1}{6}+\int_0^1\left(\frac{x^4}{1+x^4}dx-\frac{x^6}{1-x^4}dx\right)=$$

$$=\frac{1}{4}+\frac{1}{6}+\int_0^1\frac{x^4}{1+x^4}dx=\frac{1}{4}+\frac{1}{6}+\int_0^{\frac{\pi}{4}}tan^4\theta\,d\theta=\frac{r}{4}-\frac{1}{4}$$

262. Solve for real numbers:

$$\begin{cases}x^3+x^2+x=(x+1)(y+2)\sqrt{(x+1)(y+1)}\\ \sqrt{y+1}+2=\left(x-1-\frac{3}{4x}\right)\sqrt{x+1}\end{cases}$$

Proposed by Carlos Eduardo Aguiar Paiva-Brazil

Solution by proposer

$$i)\quad x^3+x^2+x=(x+1)(y+2)\sqrt{(x+1)(y+1)}$$

$$\Leftrightarrow\frac{x^3+x^2+x}{(x+1)\sqrt{x+1}}=(y+2)\sqrt{y+1}$$

$$\Leftrightarrow\frac{x^3}{(x+1)\sqrt{x+1}}+\frac{x}{\sqrt{x+1}}=(y+2)\sqrt{y+1}$$

$$\Leftrightarrow\left(\frac{x}{\sqrt{x+1}}\right)^3+\frac{x}{\sqrt{x+1}}=(y+1+1)\sqrt{y+1}=(y+1)\sqrt{y+1}+\sqrt{y+1}$$

$$\Leftrightarrow\left(\frac{x}{\sqrt{x+1}}\right)^3+\frac{x}{\sqrt{x+1}}=\left(\sqrt{y+1}\right)^3+\sqrt{y+1}$$

Let $f(m)=m^3+m$ –injective.

$$\frac{x}{\sqrt{x+1}}=\sqrt{y+1}\Rightarrow y=\frac{x^2-x-1}{x+1}$$

$$ii)\ \sqrt{y+1}+2=\left(x-1-\frac{3}{4x}\right)\sqrt{x+1}$$

$$\Leftrightarrow 4x\sqrt{y+1}+8x=(4x^2-4x-3)\sqrt{x+1}$$

$$\Leftrightarrow 4x^2+8x\sqrt{x+1}=(4x^2-4x-3)(x+1)$$

$$4x^2+8x\sqrt{x+1}+4(x+1)=(4x^2-4x+1)(x+1)$$

$$\Leftrightarrow\left(2x+2\sqrt{x+1}\right)^2=(2x-1)^2(x+1)$$

$ii')\ 2x+2\sqrt{x+1}=(2x-1)\sqrt{x+1}$ or $ii'')\ 2x+2\sqrt{x+1}=-(2x-1)\sqrt{x+1}$

$$ii')\ 4x^3-12x^2-3x+9=0\Leftrightarrow(4x^2-3)(x-3)=0\Leftrightarrow$$

$$x_{1,2}=\pm\frac{\sqrt{3}}{2};x_3=3$$

$$ii'')\ 2x+2\sqrt{x+1}=-(2x-1)\sqrt{x+1}$$

$$E=-44,F=-368,G=80,H=91584-2^6\cdot3^2\cdot159>0$$

If $H>0$, in the cubic equation, then there is only one real root:

$$x_4=\frac{-2+\sqrt[3]{10+3\sqrt{159}}+\sqrt[3]{10-3\sqrt{159}}}{6}\sim-0.233$$

$$(x,y)\in\left\{\left(\frac{\sqrt{3}}{2},\frac{4-3\sqrt{3}}{2}\right);\left(-\frac{\sqrt{3}}{2},\frac{4+3\sqrt{3}}{2}\right);\left(3,\frac{5}{4}\right)\left(\frac{-2+\sqrt[3]{10+3\sqrt{159}}+\sqrt[3]{10-3\sqrt{159}}}{6},\frac{x_4^2-x_4-1}{x_4+1}\right)\right\}$$

After the tests: $(x,y)\in\left(3,\frac{5}{4}\right)$

263. Solve for real numbers:

$$\begin{cases}x,y,z,t>0\\ \dfrac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)}=\dfrac{1}{16}\\ x+y+z+t=4\end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vişescu-Romania

If $x > 2$ then for $y > 2 \Rightarrow x + y > 4$ false. Hence, $x, y, z, t < 2$.

$$x + y = 4 - z - t = (2 - z) + (2 - t)$$

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)} = \frac{1}{16}; (2-x)(2-y)(2-z)(2-t) > 0$$

We distinguish the cases:

- $\begin{cases} 2 - x > 0 \\ 2 - y > 0 \\ 2 - z > 0 \\ 2 - t > 0 \end{cases} \Rightarrow x, y, z, t < 2$

- $\begin{cases} 2 - x > 0 \\ 2 - y > 0 \\ 2 - z < 0 \\ 2 - t < 0 \end{cases} \Rightarrow \begin{cases} z > 2 \\ t > 2 \end{cases} \Rightarrow z + t > 4$ contradiction, because $x + y + z + t = 4$

- $\begin{cases} 2 - x < 0 \\ 2 - y < 0 \\ 2 - z < 0 \\ 2 - t < 0 \end{cases} \Rightarrow x, y, z, t > 2$ contradiction, because $x + y + z + t = 4$.

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)} = \frac{1}{16} \Leftrightarrow$$

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(2-z+2-t)(2-x+2-t)(2-y+2-x)(2-y+2-z)} = \frac{1}{16}; (1)$$

$$\frac{2-z+2-t}{2} \geq \sqrt{(2-z)(2-t)} \Rightarrow \frac{1}{(2-z+2-t)} \leq \frac{1}{2\sqrt{(2-z)(2-t)}}$$

Analogously:

$$\frac{1}{(2-x+2-t)} \leq \frac{1}{2\sqrt{(2-x)(2-t)}}; \frac{1}{(2-y+2-x)} \leq \frac{1}{2\sqrt{(2-y)(2-x)}}$$

$$\frac{1}{(2-y+2-z)} \leq \frac{1}{2\sqrt{(2-y)(2-z)}}$$

Hence,

$$\frac{(2-x)(2-y)(2-z)(2-t)}{(2-z+2-t)(2-x+2-t)(2-y+2-x)(2-y+2-z)} \leq \frac{1}{16}; (2)$$

From (1), (2) it follows that: $2 - x = 2 - y = 2 - t = 2 - z \Rightarrow$

$x = y = z = t$, but $x + y + z + t = 4$ then $x = y = z = t = 1$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\begin{cases} x, y, z, t > 0 \\ \dfrac{(2-x)(2-y)(2-z)(2-t)}{(x+y)(y+z)(z+t)(t+x)} = \dfrac{1}{16}; (*) \\ x + y + z + t = 4 \end{cases}$$

If $2 - x, 2 - y \leq 0 \to 4 = x + y + z + t \leq x + y$ contradiction!

So, $(2-x)(2-y)(2-z)(2-t) = \frac{1}{16}(x+y)(y+z)(z+t)(t+x) > 0$

$\to 2 - x, 2 - y, 2 - z, 2 - t > 0.$

Let $a = 2 - x, b = 2 - y, c = 2 - z, d = 2 - t; a, b, c, d > 0; \sum a = 4$

$$\to \begin{cases} x + y = 4 - (a + b) = c + d \\ y + z = a + d \\ z + t = a + b \\ t + x = b + c \end{cases} \to (*) \Leftrightarrow (c+d)(a+d)(a+b)(b+c) = 16abcd$$

But from AM-GM we have:

$$(c+d)((a+d)(a+b)(b+c) \geq (2\sqrt{cd})(2\sqrt{ad})(2\sqrt{ab})(2\sqrt{bc}) = 16abcd$$

Equality holds when $a = b = c = d = 1 \to x = y = z = t = 1$.

264. If $x, y, z \in \mathbb{C}, x + y + z = 0$ then:

$$|x| + |y| + |z| \leq |x - z| + |z - x| + |y - x|$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Florentin Vişescu-Romania

If x, y, z –different in pairs, let $A(x), B(y), C(z)$. How $x + y + z = 0 \to \frac{x+y+z}{3} = 0 \to$

$G = O$. We must to prove that: $GA + GB + GC \leq AB + BC + CA$

$$\frac{3}{2}GA \leq AB + \frac{BC}{2}; \frac{3}{2}GB \leq BC + \frac{AC}{2}; \frac{3}{2}GC \leq CA + \frac{AB}{2}$$

$$\to \frac{3}{2}(GA + GB + GC) \leq \frac{3}{2}(AB + BC + CA)$$

If $x = y \neq z \to 2x + z = 0 \to z = -2x$

We must show that: $2|x| + |-2x| \leq |x + 2x| + |-2x - x|$ or $4|x| \leq 6|x| \Leftrightarrow$

$2|x| \geq 0$ true.

If $x = y = z \to 3x = 0 \to x = 0$.

Solution 2 by Ravi Prakash-New Delhi-India

$$a = |y - z|, b = |z - x|, c = |x - y|, AO = |x|, BO = |y|, CO = |z|$$

Let $A(x), B(y), C(z)$ be vertices of triangle ABC. Centroid G of ΔABC is given by

$$z_G = \frac{1}{3}(x + y + z) = 0; AO = \frac{2}{3}m_a = \frac{2}{3}AD$$

But $AD \le DC + AC \to \frac{3}{2}AO \le \frac{a}{2} + b$. Similarly, $\frac{3}{2}AO \le \frac{a}{2} + c$, thus $3AO \le a + b + c$.

Similarly, $3BO \le a + b + c, 3CO \le a + b + c \to$

$$3(AO + BO + CO) \le 3(a + b + c)$$

Therefore,

$$|x| + |y| + |z| \le |x - z| + |z - x| + |y - x|$$

265. Find $x, y > 0$ such that:

$$27xy + 27(1 - x - y)(x + y + xy) = 10$$

Proposed by Daniel Sitaru-Romania

Solution by Michael Sterghiou-Greece

$$27xy + 27(1 - x - y)(x + y + xy) = 10; (1)$$

Let $x + y = a > 0$, then $xy \le \frac{a^2}{4}; (1) \to$

$$0 = 27xy + 27(1 - a)(a + xy) \le 27 \cdot \frac{a^2}{4} + 2 + (1 - a)\left(a + \frac{a^2}{4}\right) =$$

$= -\frac{1}{4}(3a - 2)^2(3a + 1)$, which can only be true for $a = \frac{2}{3} \to x = \frac{2}{3} - y$

Now, (1) and $x = \frac{2}{3} - y \to -4(1 - 3y)^2 = 0 \to y = \frac{1}{3}$ and $x = \frac{1}{3}$ which are the only solution.

266. Solve for integers:

$$\sqrt[3]{a^2 + b^2} + \sqrt[3]{2ab} = a + b$$

Proposed by Mehmet Şahin-Turkey

Solution by George Florin Şerban-Romania

If $a^2+b^2=-2ab \Rightarrow (a+b)^2=0 \Rightarrow b=-a \Rightarrow \sqrt[3]{a^2+b^2}-\sqrt[3]{a^2+b^2}=0$ true.

$$\Rightarrow \{(a,-a);\, a\in\mathbb{Z}\} - \text{solution.}$$

If $a^2+b^2\neq -2ab,\, a+b\in\mathbb{Z} \Rightarrow \sqrt[3]{a^2+b^2}+\sqrt[3]{2ab}\in\mathbb{Z}$

Let us denote: $\begin{cases}\sqrt[3]{a^2+b^2}=x\\ \sqrt[3]{ab}=y\end{cases};\, x,y\in\mathbb{Z} \Rightarrow a^2+b^2+2ab=x^3+y^3$ and $x+y=a+b$.

$$(a+b)^2=x^3+y^3;\ \left(\because x^3+y^3=(x+y)(x^2-xy+y^2)\right)$$

$$\text{If } x+y=0 \Rightarrow a+b=0 \Rightarrow b=-a.$$

$$\text{If } x^2-xy+y^2-x-y=0 \Leftrightarrow x^2-(y+1)x+y^2-y=0$$

$$\Delta=-3y^2+6y+1=k^2;\, k\in\mathbb{Z}$$

$$\Delta\geq 0 \Rightarrow -3y^2+6y+1\geq 0 \Leftrightarrow 3y^2-6y-1\leq 0 \Leftrightarrow 3(y-1)^2\leq 4 \Leftrightarrow (y-1)^2\leq \frac{3}{4}$$

$$\Leftrightarrow 1-\frac{2\sqrt{3}}{3}\leq y\leq 1+\frac{2\sqrt{3}}{3};\, y\in\mathbb{Z} \Leftrightarrow y\in\{0,1,2\}$$

$$\text{If } y=0 \Rightarrow x^2-x=0 \Rightarrow x\in\{0,1\}.$$

$$x=y=0 \Rightarrow \begin{cases}a^2+b^2=0\\ 2ab=0\end{cases} \Rightarrow (a,b)\in(0,0).$$

$$\text{If } y=1 \Rightarrow x^2-2x=0 \Rightarrow x\in\{0,2\}$$

$$x=0, y=1 \Rightarrow \begin{cases}a^2+b^2=0\\ 2ab=1\end{cases} \text{ no solution.}$$

$$x=2, y=1 \Rightarrow \begin{cases}a^2+b^2=8\\ 2ab=1\end{cases} \text{ no solution.}$$

$$\text{If } x=y=2 \Rightarrow \begin{cases}a^2+b^2=8\\ 2ab=8\end{cases} \Rightarrow (a,b)=(2,2)$$

$$x=1, y=2 \Rightarrow \begin{cases}a^2+b^2=8\\ 2ab=1\end{cases} \text{ no solution.}$$

267. Solve for integers:

$$(2a^3+3a^2b+3ab^2+2b^3)(a^2+b^2)=2(a+b)^5$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by George Florin Şerban-Romania

$$2a^3+3a^2b+3ab^2+2b^3=2(a^3+b^3)+2ab(a+b)=$$

$$= 2(a+b)(a^2 - ab + b^2) + 3ab(a+b) =$$

$$= (a+b)(2a^2 - 2ab + 2b^2 + 3ab) = (a+b)(2a^2 + 2b^2 + ab)$$

$$\Rightarrow (a+b)(2a^2 + ab + 2b^2) = 2(a+b)^5$$

If $a + b = 0 \Rightarrow b = -a \Rightarrow \{(a, -a) | a \in \mathbb{Z}\}$ –solution.

If $(2a^2 + ab + 2b^2)(a^2 + b^2) = 2(a+b)^4$;

Let $a + b = s; ab = p; s, p \in \mathbb{Z}$ then:

$$(2s^2 - 4p + p)(s^2 - p) = 2s^4 \Rightarrow (2s^2 - 3p)(s^2 - 2p) = 2s^4$$

$$p(6p - 7s^2) = 0$$

If $p = 0; (ab = 0) \Rightarrow \{(a, 0); (0, b) | a, b \in \mathbb{Z}\}$ –solution.

If $6p - 7s^2 = 0 \Leftrightarrow 7(a+b)^2 = 6ab \Leftrightarrow 7a^2 + 8ab + 7b^2 = 0$

Suppose $b \neq 0 \Rightarrow 7\left(\frac{a}{b}\right)^2 + 8\left(\frac{a}{b}\right) + 7 = 0; \left(t = \frac{a}{b}\right) \Rightarrow$

$7t^2 + 8t + 7 = 0, \Delta = -132 < 0$ no solution.

Therefore,

$$S = \{(a, a), (a, 0), (0, a) | a \in \mathbb{Z}\}$$

268. Find $x, y \in \mathbb{Z}$ such that:

$$x | y \text{ and } x^2 + y^2 | y^4 + 2080$$

Proposed by Mehmet Şahin-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$x | y \Rightarrow x^2 | x^2 + y^2 | y^4 + 2080$ and $x^2 | y^4 \Rightarrow x^2 | y^4 + 2080 - y^4 = 2080 = 2^5 \cdot 5 \cdot 14$

- $x = \pm 1 \Rightarrow 1 + y^2 | y^4 + 2080 = (y^2 + 1)(y^2 - 1) + 2081$

  $\Rightarrow 1 + y^2 | 2081$, but 2081 is a prime number, then $y = 0$.

- $x = \pm 2 \Rightarrow 4 + y^2 | y^4 + 2080 = (y^2 + 4)(y^2 - 4) + 2096$

  $\Rightarrow 4 + y^2 | 2096 = 2^4 \cdot 131 \Rightarrow y^2 \in \{2^n \cdot 131^m - 4 | 0 \leq n \leq 4; 0 \leq m \leq 1\}$

  $\Rightarrow y = 0$ (others are not perfect square)

- $x = \pm 4 \Rightarrow 16 + y^2 | y^4 + 2080 = (y^2 + 16)(y^2 - 16) + 2336$

  $16 + y^2 | 2336 = 2^5 \cdot 73$

  $\Rightarrow y^2 \in \{2^n \cdot 73^m - 16 | 0 \leq n \leq 5; 0 \leq m \leq 1\}$

$y^2 = 0$ or $y^2 = 16 = 2^5 - 16$ (others are not perfect square)

$\Rightarrow y = 0$ or $y = \pm 4$

Therefore,

$$S = \{(1,0);(-1,0);(2,0);(-2,0);(4,0);(-4,0);(4,4);(4,-4);(-4,-4)\}$$

269.

$$z \in \mathbb{C}, Imz \neq 0, (z^2+1)\left(\overline{z}^2+1\right) = \left(1+\left(\frac{z+\overline{z}}{2}\right)^2\right)^2, A(i), B(-i), C(z)$$

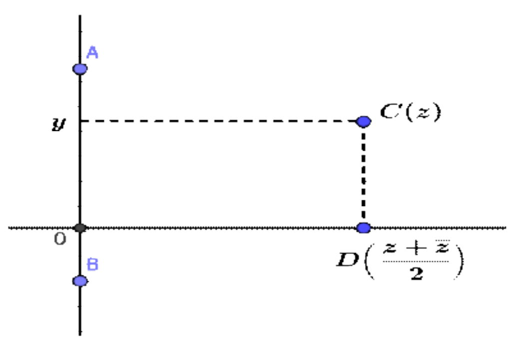
Find: $\Omega = AB + BC + CA$

Proposed by Florentin Vișescu – Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$z \in \mathbb{C}, Imz \neq 0, (z^2+1)\left(\overline{z}^2+1\right) = \left(1+\left(\frac{z+\overline{z}}{2}\right)^2\right)^2, A(i), B(-i), C(z) \quad (*)$$

Let $D\left(\frac{z+\overline{z}}{2}\right)$



$$1+\left(\frac{z+\overline{z}}{2}\right)^2 + \left(\frac{z+\overline{z}}{2}+i\right)\left(\frac{z+\overline{z}}{2}-i\right) =$$

$$= \left(\frac{z+\overline{z}}{2}+i\right)\left(\overline{\frac{z+\overline{z}}{2}+i}\right) = DB^2 = DA^2$$

$$(z^2+1)\left(\overline{z}^2+1\right) = (z+i)(z-i)(\overline{z}-i)(\overline{z}+i) =$$

$$= (z+i)\left(\overline{z+i}\right)(z-i)\left(\overline{z-i}\right) = BC^2 \times AC^2$$

$$\Rightarrow (*) \Leftrightarrow BC \times AC = DB^2 = DA^2 = 1 + OD^2$$

Let $y = Imz > 0 \Rightarrow BC > AC$

We have $OD^2 = AC^2 - (y-1)^2 = BC^2 - (y+1)^2$

$\Rightarrow BC^2 - AC^2 = 4y$ **(1)**

and $(BC - AC)^2 = OD^2 + (y+1)^2 + OD^2 + (y-1)^2 - 2 - 2OD^2$

$\Rightarrow (BC - AC)^2 = 2y^2 \Rightarrow BC - AC = \sqrt{2}y$ **(2)**

(1), (2) $\Rightarrow BC + AC = 2\sqrt{2}$ **and** $AB = 2$

$\Rightarrow \Omega = 2 + 2\sqrt{2}$ **(Similarly for** $y < 0$**)**

Solution 2 by Ravi Prakash-New Delhi-India

Let $z = x + iy, y \neq 0, (z^2 + 1)\left(\overline{z}^2 + 1\right) = \left[1 + \left(\frac{z+\overline{z}}{2}\right)^2\right]^2$

$$\Rightarrow (x^2 - y^2 + 1 + 2xyi)(x^2 - y^2 + 1 - 2xyi) = (1 + x^2)^2$$

$$\Rightarrow (x^2 - y^2 + 1)^2 + 4x^2y^2 = 1 + 2x^2 + x^4$$

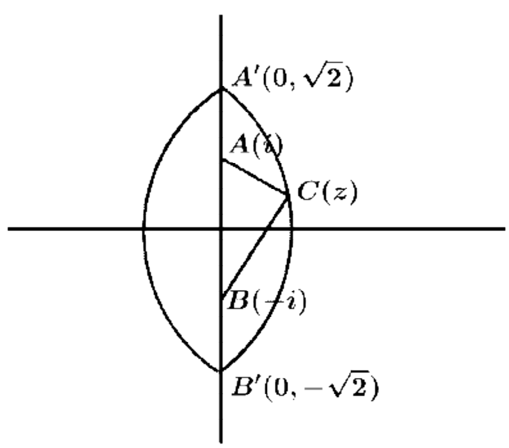
$$\Rightarrow (x^2 - y^2)^2 + 2(x^2 - y^2) + 1 + 4x^2y^2 = 1 + 2x^2 + x^4$$

$$\Rightarrow x^4 - 2x^2y^2 + y^4 - 2y^2 + 4x^2y^2 = x^4 \Rightarrow y^2(2x^2 + y^2 - 2) = 0$$

$$\Rightarrow 2x^2 + y^2 = 2 \quad [\because y \neq 0] \Rightarrow x^2 + \frac{y^2}{2} = 1$$

Eccentricity e **of the ellipse is given by** $1 = 2(1 - e^2) \Rightarrow e = \frac{1}{\sqrt{2}}$

Focii of the ellipse are $\left(0, \pm\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\right) = (0, \pm 1) = i, -i$



$$AB + AC + BC = |2i| + AA' = 2 + 2\sqrt{2} = 2(1 + \sqrt{2})$$

Solution 3 by proposer

$$z \in \mathbb{C} \setminus \mathbb{R} \Rightarrow z = a + bi, a, b \in \mathbb{R}, b \neq 0$$

$$\frac{z + \overline{z}}{2} = \frac{a + bi + a - bi}{2} = a$$

$$\Rightarrow \left(1 + \left(\frac{z + \overline{z}}{2}\right)^2\right)^2 = (1 + a^2)^2$$

$$(z^2 + 1)\left(\overline{z}^2 + 1\right) = z^2\overline{z}^2 + z^2 + \overline{z}^2 + 1 =$$

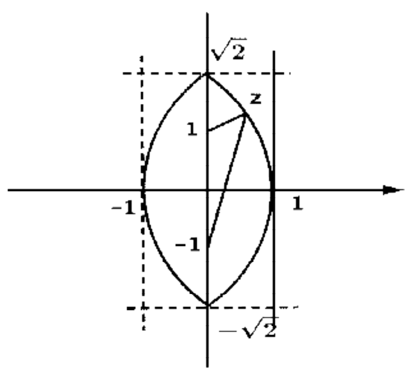
$$= |z|^4 + z^2 + \overline{z}^2 + 1 = (a^2 + b^2)^2 + a^2 + 2abi - b^2 + a^2 - 2abi - b^2 + 1 =$$

$$= a^4 + b^4 + 2a^2b^2 + 2a^2 + 1 - 2b^2$$

$$\Rightarrow a^4 + b^4 + 2a^2b^2 + 2a^2 - 2b^2 + 1 = 1 + a^4 + 2a^2$$

$$\Rightarrow b^4 + 2a^2b^2 - 2b^2 = 0| : b^2 \Rightarrow b^2 + 2a^2 - 2 = 0$$

$$2a^2 + b^2 = 2 \Rightarrow a^2 + \frac{b^2}{2} = 1 \Rightarrow \frac{a^2}{1^2} + \frac{b^2}{\sqrt{2}^2} = 1$$



So: $|z - 1| + |z + i| = 2\sqrt{2} \Rightarrow P_{\Delta ABC} = |2i| + |z - i| + |z + i| = 2 + 2\sqrt{2}$

270. $z_1, z_2, z_3 \in \mathbb{C}^*$ **different in pairs,** $|z_1| = |z_2| = |z_3| = 1$. $A(z_1), B(z_2), C(z_3)$. **If**

$$\sum_{cyc} \frac{1}{\left|(z_1 - z_2)|z_1 - z_3| + (z_1 - z_2)|z_1 - z_2|\right|^2} = \frac{3}{(|z_1 - z_2| + |z_2 - z_3| + |z_3 - z_1|)^2}$$

Then $AB = BC = CA$.

Proposed by Marian Ursărescu-Romania

Solution by proposer

Let $A(z_1), B(z_2), C(z_3) \in C(O,1)$

$$\sum_{cyc} \frac{1}{|b(z_1 - z_2) + c(z_1 - z_3)|} = \frac{3}{(a+b+c)^2} \Leftrightarrow$$

$$\sum_{cyc} \frac{(a+b+c)^2}{|z_1(b+c) - bz_2 - cz_3|^2} = 3 \Leftrightarrow \sum_{cyc} \frac{1}{\left|\frac{(a+b+c)z_1 - (az_1 + bz_2 + cz_3)}{a+b+c}\right|^2} = 3$$

$$\Leftrightarrow \sum_{cyc} \frac{1}{\left|z_1 - \frac{az_1 + bz_2 + cz_3}{a+b+c}\right|^2} = 3 \Leftrightarrow \sum_{cyc} \frac{1}{AI^2} = 3; \ (1)$$

$$AI = \frac{r}{sin\frac{A}{2}} \Rightarrow \sum_{cyc} \frac{1}{AI^2} = \frac{1}{r^2} \sum_{cyc} sin^2\frac{A}{2} = \frac{1}{r^2}\left(1 - \frac{r}{2R}\right)$$

We must to prove that:

$$\sum_{cyc} \frac{1}{AI^2} \geq \frac{3}{R^2} = 3; (2)$$

Which is true because $\left(1 - \frac{r}{2R}\right) \geq \frac{3r^2}{R^2} \Leftrightarrow \frac{3r^2}{R^2} + \frac{r}{2R} - 1 \leq 0$. Let $x = \frac{r}{R} \leq \frac{1}{2}$, hence,

$3x^2 + \frac{x}{2} - 1 \leq 0 \Leftrightarrow 6x^2 + x - 2 \leq 0 \Leftrightarrow (2x-1)(3x+2) \leq 0$ true.

From $(1), (2)$ it follows that $AB = BC = AC$.

271. $z_1, z_2, z_3 \in \mathbb{C}^*$ **different in pairs,** $|z_1| = |z_2| = |z_3|$**,** $A(z_1), B(z_2), C(z_3)$.

If

$$\sum_{cyc} \left|\frac{2z_1 - z_2 z_3}{z_2 - z_3}\right|^2 = 9 \Rightarrow AB = BC = CA.$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$A(z_1), B(z_2), C(z_3) \in C(O,R), |z_1| = |z_2| = |z_3| = R$$

$$\sum_{cyc} \left|\frac{2z_1 - z_2 z_3}{z_2 - z_3}\right|^2 = 9 \Leftrightarrow \sum_{cyc} \frac{4\left|z_1 - \frac{z_2 + z_3}{2}\right|^2}{|z_2 - z_3|} = 9 \Leftrightarrow \sum_{cyc} \frac{m_a^2}{a^2} = \frac{9}{4}; (1)$$

$$m_a^2=\frac{2(b^2+c^2)-a^2}{4}\Rightarrow\sum_{cyc}\frac{m_a^2}{a^2}=\frac{1}{4}\sum_{cyc}\frac{2(b^2+c^2)-a^2}{a^2}=$$

$$=\frac{1}{4}\left[2\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+2\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)+2\left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right)-3\right]\ge$$

$$\ge\frac{1}{4}(12-3)=\frac{9}{4};(2)$$

From (1), (2) it follows that: $AB=BC=CA$.

Solution 2 by Alex Szoros-Romania

$$\sum_{cyc}\left|\frac{2z_1-z_2z_3}{z_2-z_3}\right|^2=9\Leftrightarrow\sum_{cyc}\frac{|3z_1-(z_1+z_2+z_3)|^2}{|z_2-z_3|^2}=9$$

$$\Leftrightarrow 9\sum_{cyc}\frac{\left|z_1-\frac{z_1+z_2+z_3}{3}\right|^2}{|z_2-z_3|^2}=9\Leftrightarrow\sum_{cyc}\frac{AG^2}{BC^2}=1$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{2m_a}{3a}\right)^2=1\Leftrightarrow\sum_{cyc}\frac{m_a^2}{a^2}=\frac{9}{4}\Leftrightarrow\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4a^2}=\frac{9}{4}$$

$$\Leftrightarrow\sum_{cyc}\frac{b^2+c^2}{a^2}=6\Leftrightarrow\sum_{cyc}\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}-2\right)=0$$

$$\Leftrightarrow\sum_{cyc}\left(\frac{a}{b}-\frac{b}{a}\right)^2=0\Leftrightarrow a=b=c\Leftrightarrow AB=BC=CA.$$

Solution 3 by Ravi Prakash-New Delhi-India

Affix of G –centroid is $z_g=\frac{1}{3}(z_1+z_2+z_3)$. Also,

$$|2z_1-z_2-z_3|^2=9\left|z_1-\frac{z_1+z_2+z_3}{3}\right|^2=9AG^2=9\left(\frac{2}{3}m_a\right)^2=4m_a^2$$

$$=2(b^2+c^2)-a^2$$

$$\frac{|2z_1-z_2-z_3|^2}{|z_2-z_3|^2}=\frac{2(b^2+c^2)-a^2}{a^2}$$

Now,

$$\sum_{cyc}\frac{|2z_1-z_2-z_3|^2}{|z_2-z_3|^2}=\sum_{cyc}\left(\frac{2b^2}{a^2}+\frac{2c^2}{a^2}-1\right)\Rightarrow$$

$$2\sum_{cyc}\left(\frac{b^2}{c^2}+\frac{c^2}{a^2}-2\right)=0\Rightarrow\sum_{cyc}\left(\frac{b}{a}-\frac{a}{a}\right)^2=0\Rightarrow a=b=c\Rightarrow AB=BC=CA.$$

272. If $x, y\in\mathbb{C}$ then

$$|x|+|y|+|3x+2y|\le|4x+3y|+2|x+y|+|y-x|$$

Proposed by Daniel Sitaru-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have:

$$|4x+3y|+|x+y|\overset{\Delta}{\geq}|(4x+3y)-(x+y)|=|3x+2y|;(1)$$

$$|x+y|+|y-x|\overset{\Delta}{\geq}|(x+y)+(y-x)|=2|y|;(2)$$

$$|x+y|+|y-x|\overset{\Delta}{\geq}|(x+y)-(y-x)|=2|x|;(3)$$

From $(1),(2),(3)$ it follows that:

$$|x|+|y|+|3x+2y|\le|4x+3y|+2|x+y|+|y-x|$$

273. Find all numbers: $\Omega=\overline{abcd}$ such that:

$$\sqrt{(b+d)^2-a^2+4c-4}+2(b^2+d^2)+c^2-4a+4=0$$

Proposed by George Florin Șerban-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{(b+d)^2-a^2+4c-4}+2(b^2+d^2)+c^2-4a+4\overset{(*)}{=}0$$

(*) is defined when $(b+d)^2-a^2+4c-4\ge0\Leftrightarrow(b+d)^2\ge a^2-4c+4$

We have: $2(b^2+d^2)\ge(b+d)^2\leftrightarrow(b-d)^2\ge0$

Which is true with equality holds when $b=d\rightarrow2(b^2+d^2)\ge a^2-4c+4$

$$\rightarrow2(b^2+d^2)+c^2-4a+4\ge a^2-4c+4+c^2-4a+4=(a-2)^2+(c-2)^2\ge0$$

$\rightarrow\sqrt{(b+d)^2-a^2+4c-4}+2(b^2+d^2)+c^2-4a+4\ge0$**, equality holds when:**

$$\sqrt{(b+d)^2-a^2+4c-4}=2(b^2+d^2)+c^2-4a+4=0$$

$\leftrightarrow b=d, a=c=2$ **and** $(b+d)^2=a^2-4c+4\rightarrow a=c=2$ **and** $b=d=0$

Therefore, $\Omega=\overline{abcd}=2020.$

274. Solve in $\mathbb{R}$ the following equation:

$$2\cdot 3^x+5\cdot 4^x=4\cdot 5^x+3\cdot 2^x$$

Proposed by Florentin Vișescu – Romania

Solution by proposer

We write the equation: $2\cdot 3^x-3\cdot 2^x=4\cdot 5^x-5\cdot 4^x$

We consider the function $f:\mathbb{R}\to\mathbb{R}, f(t)=t^x$

On the intervals $[2,3]$ and $[4,5]$ f meets the conditions of Pompeiu's Theorem.

$$\left.\begin{array}{c} f:[a,b]\to\mathbb{R} \\ f\ continuous\ on\ [a,b] \\ f\ derivable\ on\ (a,b) \\ 0\notin[a,b]\end{array}\right\}\Rightarrow \exists c\in(a,b)$$

$$\frac{af(b)-bf(a)}{a-b}=f(c)-cf'(c)$$

So, $\exists c_1\in(2,3)$ such that $\frac{2f(3)-3f(2)}{2-3}=f(c_1)-c_1f'(c_1)$

$$\Rightarrow\frac{2\cdot 3^x-3\cdot 2^x}{-1}=c_1^x-c_1\times c_1^{x-1}=c_1^x-xc_1^x$$

$\exists c_2\in(4,5)$ such that $\frac{4\cdot 5^x-5\cdot 4^x}{-1}=c_2^x-xc_2^x$

Then, we obtain $c_1^x(1-x)=c_2^x(1-x)$ or

$$(1-x)(c_1^x-c_2^x)=0$$

$$1-x=0\Rightarrow x=1,\left(\frac{c_1}{c_2}\right)^x=1\Rightarrow x=0, S=\{0,1\}$$

275. Find $x,y,z\geq 1$ such that:

$$\begin{cases} x^3+y^2+2z^2=4 \\ 729\cdot\prod_{cyc}(\log(xy)\cdot\log z)=8\cdot\log^6(xyz)\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Asmat Qatea-Afghanistan

$\prod_{cyc}\ln(z)\leq\frac{1}{27}\ln^3(xyz)\to$ by AM-GM

$$\prod_{cyc}\ln(xy)\le\frac{8}{27}\ln^3(xyz)\to \text{by AM-GM}$$

$$729\prod_{cyc}\ln(xy)\cdot\ln(z)\le 8\ln^6(xyz)$$

Equality holds when $x=y=z$

$$x^3+y^2+2z^2=4,\qquad x^3+3x^2=4\Rightarrow x^3+3x^2-4=0$$

$$x^3-x^2+4x^2-4=0,\qquad x^2(x-1)+4(x-1)(x+1)=0$$

$$(x-1)(x^2+4x+4)=0,\Delta=16-16=0$$

$$x_1=1,x_{1,2}=\frac{-4\pm 0}{2}=-2.\ \text{Solution } x=y=z=1$$

276. Solve for integers:

$$(x^2+y^2)(x^4+y^4)=(x+y)^6$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$(x^2+y^2)(x^4+y^4)=(x+y)^6;\ \ (*)$$

$$\text{Let } a=x^2+y^2,b=2xy\Rightarrow(*)\Leftrightarrow a(2a^2-b^2)=2(a+b)^3$$

$$6a^2b+7ab^2+2b^3=0\Leftrightarrow b(2a+b)(3a+2b)=0$$

$$\Leftrightarrow b=0\text{ or }2a+b=0\text{ or }3a+2b=0$$

$$\Leftrightarrow x=0\text{ or }y=0\text{ or }x^2+y^2+(x+y)^2=0\text{ or }x^2+y^2+2(x+y)^2=0$$

$$\Leftrightarrow x=0\text{ or }y=0.$$

Therefore,

$$S=\{(x,0);(0y,y)|x,y\in\mathbb{Z}\}$$

Solution 2 by George Florin Şerban-Romania

$$\text{Let } x+y=s;xy=p\Rightarrow x^2+y^2=(x+y)^2-2xy=s^2-2p$$

$$x^4+y^4=(x^2+y^2)^2-2x^2y^2=(s^2-2p)^2-2p^2=s^4-4s^2p+2p^2$$

$$(x^2+y^2)(x^4+y^4)=(x+y)^6\Leftrightarrow(s^2-2p)(s^4-4s^2p+2p^2)=s^6$$

$$p(-6s^4+10s^2p-4p^2)=0$$

If $p=0\Rightarrow xy=0\Rightarrow x=0$ or $y=0\Rightarrow S=\{(x,0);(0y,y)|x,y\in\mathbb{Z}\}$ solution.

$$\text{If }-6s^4+10s^2p-4p^2=0\Rightarrow(s^2-p)(6s^2-4p)=0$$

If $s^2 - p = 0 \Rightarrow (x+y)^2 = xy \Rightarrow x^2 + xy + y^2 = 0$

$\left(\frac{x}{y}\right)^2 + \frac{x}{y} + 1 = 0; t = \frac{x}{y} \Rightarrow t^2 + t + 1 = 0$ –no has solution.

If $6s^2 - 4p = 0 \Rightarrow 3(x+y)^2 = 2xy \Rightarrow 3x^2 + 4xy + 3y^2 = 0$ no has solution.

Therefore,

$$S = \{(x,0);(0,y)|x,y \in \mathbb{Z}\}$$

277. Solve for integers:

$$\sqrt{x^3+y^3} + \sqrt{y^3+z^3} + \sqrt{z^3+x^3} = 2(x+y+z)$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sqrt{x^3+y^3} + \sqrt{y^3+z^3} + \sqrt{z^3+x^3} = 2(x+y+z);(*)$$

$$x^3+y^3 = (x+y)\left[\left(x-\frac{1}{2}y\right)^2 + \frac{3}{4}y^2\right] \geq 0 \Rightarrow x+y \geq 0, y+z \geq 0, z+x \geq 0$$

$$\sqrt{x^3+y^3} - (x+y) = \sqrt{x+y}\left(\sqrt{x^2-xy+y^2} - \sqrt{x+y}\right) =$$

$$= \sqrt{x+y} \cdot \frac{x^2-(y+1)x+y^2-y}{\sqrt{x^2-xy+y^2}+\sqrt{x+y}}$$

$$x^2-(y+1)x+y^2-y = \left(x-\frac{y+1}{2}\right)^2 + \frac{1}{4}(3y^2-6y-1) =$$

$$= \left(x-\frac{y+1}{2}\right)^2 + \frac{3}{4}\left(y-\frac{3-2\sqrt{3}}{3}\right)\left(y-\frac{3+2\sqrt{3}}{3}\right)$$

Hence, if $y \in \mathbb{Z} - \{0,1,2\}; \sqrt{x^3+y^3} > x+y, \forall x \in \mathbb{Z}; (1)$

Using (1), if $x, y \in \mathbb{Z} - \{0,1,2\}$ it follows that $\sqrt{x^3+y^3} > x+y$,

$\sqrt{y^3+z^3} > y+z; \sqrt{z^3+x^3} > z+x$

$\Rightarrow \sqrt{x^3+y^3} + \sqrt{y^3+z^3} + \sqrt{z^3+x^3} > 2(x+y+z)$

So, it is necessary that two of x, y, z be in $\{0,1,2\}$; y for example.

$z = 0: (*) \Leftrightarrow \sqrt{x^3+y^3} + \sqrt{y^3} + \sqrt{x^3} = 2(x+y), x \geq 0$

$y = 0: (*) \Leftrightarrow \sqrt{x^3} = x \Leftrightarrow x = 1$ or $x = 0$.

$y = 1: (*) \Leftrightarrow \sqrt{x^3+1} + \sqrt{x^3} = 2x+1$

$x > 2 \Rightarrow \sqrt{x^3+1} \overset{(1)}{\geq} x+1$ and $\sqrt{x^3} > x \Rightarrow \sqrt{x^3+1} + \sqrt{x^3} > 2x+1 \Rightarrow$

$x \in \{0,1,2\} \Rightarrow x = 0$

If $y = 2(*) \Leftrightarrow \sqrt{x^3 + 2^3} + \sqrt{x^3} + 2\sqrt{2} = 2(x+2)$

Similarly, using $(1) \Rightarrow x \in \{0,1,2\} \Rightarrow (*)$ does not admit a solution.

$\Rightarrow z = 0 \Rightarrow (x = 0, y = 0)$ or $(x = 1, y = 0)$ or $(x = 0, y = 1)$

If $z = 1 \Rightarrow (*) \Leftrightarrow \sqrt{x^3 + y^3} + \sqrt{y^3 + z^3} + \sqrt{z^3 + x^3} = 2(x + y + z), x \ge -1$

$y = 0 \Rightarrow x = 0$ (similarly to $z = 0$ and $y = 1$)

$$y = 1 \Rightarrow (*) \Leftrightarrow 2\sqrt{x^3 + 1} + \sqrt{2} = 2(x + 2); x \ge 1$$

If $x \ge 3 \Rightarrow x^3 + 1 \ge 3x^2 + 1 \overset{x\ge 3}{\ge} x^2 + 6x + 1 \overset{x \ge 3}{\ge} x^2 + 4x + 7 > (x+2)^2$

$\Rightarrow 2\sqrt{x^3+1} + \sqrt{2} > 2(x+2) \Rightarrow x \in \{-1,0,1,2\} \Rightarrow (*)$ does not admit a solution.

$$y = 2 \Rightarrow (*) \Leftrightarrow \sqrt{x^3 + 2^3} + \sqrt{x^3 + 1} = 2x + 3, x \ge -1$$

Using $(1) \Rightarrow x \in \{0,1,2\} \Rightarrow x = 2 \Rightarrow z = 1 \Rightarrow (x = y = 0)$ or $(x = y = 2)$.

If $z = 2$: $(*) \Leftrightarrow \sqrt{x^3 + y^3} + \sqrt{y^3 + 8} + \sqrt{8 + x^3} = 2(x + y + 2), x \ge -2$

$y = 0 \Rightarrow (*)$ does not admit solution (similarly to $z = 0$ and $y = 2$)

$y = 1 \Rightarrow x = 2$ (similarly to $z = 1$ and $y = 2$)

$$y = 2 \Rightarrow (*) \Leftrightarrow \sqrt{x^3 + 2^3} = x + 2, x \ge -2$$

By (1), $x \in \{0,1,2\} \Rightarrow x = 1$ or $x = 2$.

$\Rightarrow z = 2 \Rightarrow (x = y = 2)$ or $(x = 1, y = 2)$ or $(x = 2, y = 1)$.

$$S = \{(0,0,0);(0,0,1);(0,1,0);(1,0,0),(1,2,2);(2,1,2);(2,2,2)\}$$

278. Solve for integers:

$$x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy} = 3xyz$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let S be the set of solutions of $(*)$.

$$x = 0 \to 0 = 0, (\forall y, z \in \mathbb{Z}, yz \ge 0)$$

$$\to \{(0, y_1, z_1), (x_2, 0, z_2), (x_3, y_3, 0) \mid (x_2, y_1, y_3, z_1, z_2) \in \mathbb{Z}^5; y_1 z_1, x_2 z_2, x_3 y_3 \ge 0\} \in S$$

Now, we assume that $x, y, z \in \mathbb{Z}^*$. We have: $xy, yz, zx > 0$ and

$$3xyz = x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy} > 0 \to x, y, z > 0$$

By AM-GM we have:

$$3xyz = x^2\sqrt{yz} + y^2\sqrt{zx} + z^2\sqrt{xy} \ge 3\sqrt[8]{\prod_{cyc}\left(x^2\sqrt{yz}\right)} = 3xyz.$$

Equality holds when $x^2\sqrt{yz} = y^2\sqrt{zx} = z^2\sqrt{xy} \Leftrightarrow x\sqrt{x} = y\sqrt{y} = z\sqrt{z} \Leftrightarrow x = y = z.$

Finally,

$$S = \left\{(0, y_1, z_1), (x_2, 0, z_2), (x_3, y_3, 0) \middle| \begin{matrix}(x_2, y_1, y_3, z_1, z_2) \in \mathbb{Z}^5; x_1 \in \mathbb{N} \\ y_1z_1, x_2z_2, x_3y_3 \ge 0\end{matrix}\right\}$$

279. Find all $(x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that:

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 2020$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 2020; (*)$ **–is symmetric, then we can assume that** $x \le y \le z$.

It is clear that $(0, 0, z)$ **is a solution of** $(*)$ **for all** $a \in \mathbb{N}$. **Assume that:** $y > 0$.

We have: $3x \le \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 2020 \to x \le 673.$

Now, we fix $x \le 673$, **we have:** $2\sqrt{xy} + y \le \sqrt{xy} + \sqrt{yz} + \sqrt{zx} \le 2020.$

$$\to \sqrt{y^2} + 2\sqrt{x}\sqrt{y} - 2020 \le 0, \Delta = 4(x + 2020) > 0$$

$$\to \sqrt{y} \le \sqrt{x + 2020} - \sqrt{x} \to x \le y \le \left[\left(\sqrt{x + 2020} - \sqrt{x}\right)^2\right]$$

Now, we fix $x \le 673, x \le y \le \left[\left(\sqrt{x + 2020} - \sqrt{x}\right)^2\right] \to \sqrt{z} \le \frac{2020-\sqrt{xy}}{\sqrt{x}+\sqrt{y}}$

$$y \le z \le \left[\left(\frac{2020 - \sqrt{xy}}{\sqrt{x} + \sqrt{y}}\right)^2\right]$$

$$S = \left\{\begin{matrix}(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a), (d, 0, 0), (0, d, 0), (0, 0, d) | \\ a \le 673, a \le b \le \left[\left(\sqrt{a + 202} - \sqrt{a}\right)^2\right], b \le c \le \left[\left(\frac{2020 - \sqrt{ab}}{\sqrt{a} + \sqrt{b}}\right)^2\right], d \in \mathbb{N}\end{matrix}\right\}$$

280. Solve for natural numbers:

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}=(x+y+z)\sqrt{\frac{x+y+z}{3}}$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have: $x\sqrt{y}+y\sqrt{z}+z\sqrt{x}=\sqrt{x}\sqrt{xy}+\sqrt{y}\sqrt{yz}+\sqrt{z}\sqrt{zx}\overset{CBS}{\leq}$

$$\leq\sqrt{x+y+z}\cdot\sqrt{xy+yz+zx}\overset{3\sum xy\leq(\sum x)^2}{\leq}\sqrt{x+y+z}\cdot\sqrt{\frac{1}{3}(x+y+z)^2}$$

Hence,

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}=(x+y+z)\sqrt{\frac{x+y+z}{3}}$$

Equality holds when: $3(xy+yz+zx)=(x+y+z)^2\leftrightarrow$

$$(x-y)^2+(y-z)^2+(z-x)^2=0\leftrightarrow x=y=z$$

$$S=\{(x,x,x)|x\in\mathbb{N}\}$$

281. Solve for integers:

$$x^3+y^3+z^3=\sqrt[3]{xyz}(x^2+y^2+z^2)$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by Dang Le Minh Nhat-Vietnam

Let $p=x+y+z,q=xy+yz+zx,r=xyz$. We have:

$$\sqrt[3]{xyz}(x^2+y^2+z^2)\leq\frac{x+y+z}{3}\cdot(x^2+y^2+z^2)=\frac{p^3-2pq}{3}.$$

We will prove: $x^3+y^3+z^3=p^3-3pq+3r\geq\frac{p^3-2pq}{3}\rightarrow$

$$2p^3-7pq+9r\geq0 \text{ and } 9r\geq4pq-p^3(Schur's)$$

Then: $2p^3-7pq+9r\geq2p^3-7pq+4pq-p^3=p^3-3pq.$

We prove: $p^3-3pq\geq0\rightarrow p^2\geq3q$, which is true.

Equality occurs since $x = y = z, x, y, z \in \mathbb{Z}$.

282.

$$a, b, c, d > 0, \frac{ab + bc + cd + da}{a + b + c + d} = \frac{ab}{a + b} + \frac{cd}{c + d} = \frac{ad}{a + d} + \frac{bc}{b + c}$$

Prove that: $(bd + a^2)(bd + c^2) = (ac + b^2)(ac + d^2)$

Proposed by Daniel Sitaru – Romania

Solution by Mohammed Diai-Rabat-Morocco

$$(bd + a^2)(bd + c^2) = (ac + b^2)(ac + d^2) \quad (*)$$

We have: (*) $\Leftrightarrow bd(a^2 + c^2) = ac(b^2 + d^2) \Leftrightarrow bd(a - c)^2 = ac(b - d)^2$ (**)

$$\frac{ab}{a + b} + \frac{cd}{c + d} = \frac{ad}{a + d} + \frac{bc}{b + c} \Leftrightarrow \frac{ab(c + d) + cd(a + b)}{(a + b)(c + d)} = \frac{ad(b + c) + bc(a + d)}{(a + d)(b + c)}$$

$$\Leftrightarrow \frac{abc + abd + acd + bdc}{(a + b)(c + d)} = \frac{abd + acd + abc + bcd}{(a + d)(b + c)}$$

$$\Leftrightarrow (a + b)(c + d) = (a + d)(b + c) \Leftrightarrow ad + bc = ab + cd$$

$$\Leftrightarrow (a - c)(b - d) = 0 \Leftrightarrow a = c \text{ or } b = d$$

Case: $a = c$ (The other case is similar)

To prove (**) we must prove that: $a = c \Rightarrow b = d$

$$\frac{ab + bc + cd + da}{a + b + c + d} = \frac{ab}{a + b} + \frac{cd}{c + d} \Leftrightarrow \frac{2a(b + d)}{2a + b + d} = a\left(\frac{b}{a + b} + \frac{d}{a + d}\right)$$

$$\Leftrightarrow (2b + 2d)(a^2 + ab + ad + bd) = (2a + b + d)(ab + ad + 2bd)$$

$$\Leftrightarrow a(b - d)^2 = 0 \Leftrightarrow b = d \text{ Q.E.D.}$$

283. φ –Euler's totient function, $\sigma(n) = \sum_{(d|n)} d$

Find:

$$\Omega = \lim_{x \to \infty} \frac{\varphi(n) \cdot \sigma(n)}{n^3}$$

Proposed by Adi Abdullayev-Baku-Azerbaijan

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $n=\prod_{i=1}^{k}p_i^{\alpha_i}$ be the factorization of n, $n>1$.

We have: $\sigma(n)=\prod_{i=1}^{k}\frac{p_i^{\alpha_i+1}-1}{p_i-1}$ and $\phi(n)=n\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right)=n\prod_{i=1}^{k}\frac{p_i-1}{p_i}$

$$\phi(n)\sigma(n)=\left(n\prod_{i=1}^{k}\frac{p_i-1}{p_i}\right)\left(\prod_{i=1}^{k}\frac{p_i^{\alpha_i+1}-1}{p_i-1}\right)=n\prod_{i=1}^{k}\frac{p_i^{\alpha_i+1}-1}{p_i}=$$

$$=n\prod_{i=1}^{k}\left(p_i^{\alpha_i}-\frac{1}{p_i}\right)<n\prod_{i=1}^{n}p_i^{\alpha_i}=n^2$$

$$0\le\frac{\phi(n)\sigma(n)}{n^3}<\frac{1}{n}\to 0(n\to\infty)$$

Therefore, $\Omega=0$.

Solution 2 by Amrit Awasthi-India

It is well known: $\frac{6}{\pi^2}<\frac{\phi(n)\sigma(n)}{n^2}\le 1|\cdot\frac{1}{n}\to\frac{6}{n\pi^2}<\frac{\phi(n)\sigma(n)}{n^3}\le\frac{1}{n}$

$$0\le\frac{\phi(n)\sigma(n)}{n^3}<\frac{1}{n}\to 0(n\to\infty)$$

Therefore, $\Omega=0$.

284. Find all $n\in\mathbb{N}$ such that:

$$\prod_{k=0}^{n}\frac{(k^2+k+1)^2+1}{(k^2+1)^2}=\left[\frac{n}{7}\right]+2020,[*]-GIF$$

Proposed by George Florin Șerban-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{(k^2+k+1)^2+1}{(k^2+1)^2}=\frac{k^4+2k^3+3k^2+2k+2}{(k^2+1)^2}=\frac{(k^2+1)(k^2+2k+2)}{(k^2+1)^2}$$

$$=\frac{(k+1)^2+1}{k^2+1}$$

$$\to\prod_{k=0}^{n}\frac{(k^2+k+1)^2+1}{(k^2+1)^2}=\prod_{k=0}^{n}\frac{(k+1)^2+1}{k^2+1}=\frac{(n+1)^2+1}{0^2+1}=n^2+2n+2$$

$$\prod_{k=0}^{n}\frac{(k^2+k+1)^2+1}{(k^2+1)^2}=\left[\frac{n}{7}\right]+2020\leftrightarrow n^2+2n-2018=\left[\frac{n}{7}\right]$$

$\to \frac{n}{7}-1<n^2+2n-2018\le\frac{n}{7}\leftrightarrow 7n^2+13n-14119>0$ and $7n^2+13n-14126\le 0$

$$7n^2+13n-14119>0\to n\ge\frac{-13+\sqrt{13^2+4\cdot 7\cdot 14119}}{14}>43$$

$$7n^2+13n-14126\le 0\to n\le\frac{-13+\sqrt{13^2+4\cdot 7\cdot 14126}}{14}<45$$

Therefore, $n=44$

Solution 2 by Samar Das-India

$$\prod_{k=0}^{n}\frac{(k^2+k+1)^2+1}{(k^2+1)^2}=\left[\frac{n}{7}\right]+2020 \quad \text{<where } n\in\mathbb{N}\text{>}$$

$$\Rightarrow\prod_{k=0}^{n}\frac{(k^2+1)^2+2k(k^2+1)+(k^2+1)}{(k^2+1)^2}=\left[\frac{n}{7}\right]+2020$$

$$\Rightarrow\prod_{k=0}^{n}\frac{(k^2+1)(k^2+1+2k+1)}{(k^2+1)^2}=m+2020$$

$$\Rightarrow\prod_{k=0}^{n}\frac{k^2+2k+2}{k^2+1}=m+2020\ \left(m=\left[\frac{n}{7}\right]\text{ which is an integer}\right)$$

$$\Rightarrow\frac{2}{1}\times\frac{5}{2}\times\frac{10}{5}\times\dots\times\frac{n^2+1}{(n-1)^2+1}\times\frac{n^2+2n+2}{n^2+1}=m+2020$$

$$\Rightarrow\frac{n^2+2n+2}{1}=m+2020\quad (1)$$

$$\because\frac{n}{7}-1\le m\le\frac{n}{7}\overset{(1)}{\Rightarrow}\frac{n}{7}-1\le n^2+2n-2018\le\frac{n}{7}\quad (2)$$

when $n^2+2n-2018\le\frac{n}{7}\Rightarrow n^2+\frac{13}{7}n-2018\le 0$

$$\Rightarrow\left(n+\frac{13}{14}\right)^2-\left(2018+\frac{169}{196}\right)\le 0\Rightarrow\left(n+\frac{13}{14}\right)^2-\left(\frac{\sqrt{395697}}{14}\right)^2\le 0\quad (3)$$

$$\Rightarrow-\frac{\sqrt{395697}}{14}\le\frac{n}{7}+\frac{13}{14}\le\frac{\sqrt{395697}}{14}\Rightarrow-\left(\frac{13}{14}+\frac{\sqrt{395697}}{14}\right)\le n\le\frac{\sqrt{395697}}{14}-\frac{13}{14}$$

$\Rightarrow -45.86\le n\le 44.0032\ \therefore n=44$ (where is positive integers)

And when $n^2+2n-2018\ge\frac{n}{7}-1$

$$\Rightarrow n^2+n\times\frac{13}{7}-2017\ge 0\Rightarrow\left(n+\frac{13}{14}\right)^2-\left(\frac{\sqrt{395501}}{14}\right)^2\ge 0\quad (4)$$

So, $n + \frac{13}{14} - \frac{\sqrt{395501}}{14} \geq 0 \Rightarrow n \geq 43.99 \Rightarrow n = 44$

But, $n \leq -\frac{13}{14} - \frac{\sqrt{395501}}{14}$ is not possible, since n is positive integers

285. **Solve for real numbers:**

$$\frac{\sin(\sin x)}{\sin x} + \frac{\cos(\cos x)}{\cos x} = 1$$

Proposed by Rovsen Pirguliyev-Sumgait-Azerbaijan

Solution by Mohammed Diai-Rabat-Morocco

$$\frac{\sin(\sin x)}{\sin x} + \frac{\cos(\cos x)}{\cos x} = 1 \quad (*)$$

First domain of solutions is: $D = \mathbb{R} \setminus \left\{k\pi, \frac{\pi}{2} + k'\pi\right\}\ k, k' \in \mathbb{Z}$

(*) $\Leftrightarrow \frac{\cos(\cos x)}{\cos x} = \frac{\sin x - \sin(\sin x)}{\sin x} > 0$ because $\begin{cases}\sin x > 0 \Rightarrow \sin x > \sin(\sin x)\\ \sin x < 0 \Rightarrow \sin x < \sin(\sin x)\end{cases}$

And since $\cos x \in]-1,1[\subset \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ therefore $\cos(\cos x) > 0$

So we conclude that $\cos x > 0 \Rightarrow x \in \left]-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right[, k \in \mathbb{Z}$

From now we will study the domain $D_* = \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

Let be the function f defined by $f(x) = \frac{\sin x}{x}$, $f'(x) = \frac{\cos x(x - \tan x)}{x^2}$

x	$-\frac{\pi}{2}$		-1		0		1		$+\frac{\pi}{2}$
$f'(x)$		$+$		$+$		$-$		$-$	
$f(x)$	$\frac{2}{\pi}$	$\nearrow$	$\sin 1$	$\nearrow$	1	$\searrow$	$\sin 1$	$\searrow$	$\frac{2}{\pi}$

Therefore: $\sin 1 < \frac{\sin(\sin x)}{\sin x} < 1$ (**)

Let be the function g defined by $g(x) = \frac{\cos x}{x}$, $g'(x) = -\frac{x\sin x + \cos x}{x^2} = -\frac{h(x)}{x^2}$

$h'(x) = \sin x + x\cos x - \sin x = x\cos x$ then

x	$-\frac{\pi}{2}$		-1		0		1		$+\frac{\pi}{2}$
$h'(x)$		$-$		$-$	0	$+$		$+$	
$h(x)$	$\frac{\pi}{2}$	$\searrow$		$\searrow$	1	$\nearrow$		$\nearrow$	$\frac{\pi}{2}$
$g'(x)$		$-$		$-$	$\|\|$	$-$		$-$	
$g(x)$	0	$\searrow$	$-\cos 1$	$\searrow$	$-\infty \|\| +\infty$	$\searrow$	$\cos 1$	$\searrow$	0

$$\cos x \in]0,1[\Rightarrow \frac{\cos(\cos x)}{\cos x} > \cos 1$$

Using (**) $1 = \frac{\sin(\sin x)}{\sin x} + \frac{\cos(\cos x)}{\cos x} > \sin 1 + \cos 1$ impossible since $\sin 1 + \cos 1 > 1$

Conclusion the equation (*) have no real solutions.

286. Let $n \in \mathbb{N}, n \geq 1$. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(2020x^{2n}) = f(-2020y^{2n}) + x^{2n}y^{2n+1}, \forall x, y \in \mathbb{R}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

Solution by Mohammed Diai-Rabat-Morocco

$$f(2020x^{2n}) = f(-2020y^{2n}) + x^{2n}y^{2n+1}, \forall x, y \in \mathbb{R}; (*)$$

Let f –an eventual solution. If f –is constant solution, then $\exists c \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) = c$

$$(*) \to \forall x, y \in \mathbb{R}: x^{2n}y^{2n+1} = 0,$$

which is not true, hence there is no constant solution for $(*)$.

Let be $a \geq 0$. Taking $y = 0$ in $(*)$: $\forall x \in \mathbb{R}, f(2020x^{2n}) = f(0)$; $(**)$

Taking $x = \left(\frac{a}{2020}\right)^{\frac{1}{2n}}$ in $(**)$: $f(a) = f(0)$. Therefore, $\forall a \in \mathbb{R}_+$: $f(a) = f(0)$; (1)

Let be $a \leq 0$. Taking $x = 0$ in $(*)$: $\forall y \in \mathbb{R}, f(-2020y^{2n}) = f(0)$; $(***)$

Taking $y = \left(-\frac{a}{2020}\right)^{\frac{1}{2n}}$ in $(***)$: $f(a) = f(0)$. Therefore, $\forall a \in \mathbb{R}_-, f(a) = f(0)$; (2)

From $(1), (2)$ it follows that f –is a constant function on $\mathbb{R}$ which is impossible.

Therefore, there is no solution function satisfying $(*)$

287. Find all n – prime numbers such that

$$|n^3 - 3n^2 + 2n - 203|, |n^4 - 18n^3 + 199n^2 - 342n + 325|$$

are prime numbers.

Proposed by George Florin Șerban-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $A(n) = |n^3 - 3n^2 + 2n - 203|, B(n) = |n^4 - 18n^3 + 119n^2 - 342n + 325|$

We have: $A(7) = 7$ a prime number and $B(7) = 11$ a prime number, so $n = 7$ is a solution.

If $n \neq 7 \rightarrow n \equiv 1, 2, 3, 4, 5, 6$ (mod 7) because n is a prime number.

We have: $A(n) = |n(n-1)(n-2) - 203|$ and $203 \equiv 0$ (mod 7)

$\rightarrow$ If $n \equiv 1, 2 (mod\ 7) \rightarrow (n-1)(n-2) \equiv 0 (mod\ 7) \rightarrow A(n) \equiv 0\ (mod\ 7)$

$A(n) = 7 \leftrightarrow n(n-1)(n-2) = 210$ or $n(n-1)(n-2) = 196$

$$n(n-1)(n-2) = 210 = 7.6.5 \rightarrow n = 7$$

$n(n-1)(n-2) = 196$ has no solution on IN because $n(n-1)(n-2) \equiv 0 (mod\ 3)$ but $196 \equiv 1 (mod\ 3)$

So, if $n \equiv 1, 2 (mod\ 7) \rightarrow A(n)$ is not a prime number.

We also have: $B(n) = |(n-6)(n-5)(n-4)(n-3) - 35|$ and $35 \equiv 0\ (mod\ 7)$

$\rightarrow$ If $n = 3, 4, 5, 6 (mod\ 7) \rightarrow (n-6)(n-5)(n-4)(n-3) \equiv 0 (mod\ 7) \rightarrow B(n) \equiv 0 (mod\ 7)$

$B(n) = 7 \leftrightarrow (n-6)(n-5)(n-4)(n-3) = 42$ or $(n-6)(n-5)(n-4)(n-3) = 28$

$(n-6)(n-5)(n-4)(n-3) = 42$ has no solution on IN because:

$(n-6)(n-5)(n-4)(n-3) \equiv 0 (mod\ 4)$ but $42 \equiv 2 (mod\ 4)$

$(n-6)(n-5)(n-4)(n-3) = 28$ has no solution on IN because:

$(n-6)(n-5)(n-4)(n-3) = 0 (mod\ 3)$ but $28 \equiv 1 (mod\ 3)$

So, if $n \equiv 3, 4, 5, 6 (mod\ 7) \rightarrow B(n)$ is not a prime number.

Finally, the only solution is $n = 7$.

288. Solve for real numbers: $\begin{cases} x^2 = yz + 1 \\ y^2 = zx + a \\ z^2 = xy + a^2 \end{cases}, a > 1$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohammed Diai-Rabat-Morocco

$$\begin{cases} x^2 = yz + 1 \ (1) \\ y^2 = zx + a \ (2) \\ z^2 = xy + a^2 (3) \end{cases}, a > 1$$

(1)–(2)$\Rightarrow (x - y)(x + y) = z(y - x) + 1 - a \Rightarrow (y - x)(x + y + z) = a - 1 > 0$ **(3)**

(1)–(3)$\Rightarrow (x - z)(x + z) = y(z - x) + 1 - a^2 \Rightarrow (z - x)(x + y + z) = a^2 - 1 > 0$ **(4)**

The real numbers x, y, z **must be different since** $a > 1$

$$(4)/(3) \Rightarrow \frac{z-x}{y-x} = a + 1 \Rightarrow ax - (a + 1)y + z = 0 \Rightarrow z = (a + 1)y - ax \quad \textbf{(5)}$$

(5) and (3)$\Rightarrow \left((a + 1)y - ax\right)^2 = xy + a^2 \Rightarrow$

$$\Rightarrow a^2x^2 + (a + 1)^2y^2 - ((a + 1)^2 + a^2)xy = a^2$$

$$\Rightarrow a^2x(x - y) + (a + 1)^2y(y - x) = a^2 \Rightarrow (x - y)(a^2x - (a + 1)^2y) = a^2 \quad \textbf{(6)}$$

(1)+(2) and (5)$\Rightarrow x^2 + y^2 = \left((a + 1)y - ax\right)(x + y) + 1 + a \Rightarrow$

$$\Rightarrow (a + 1)x^2 - ay^2 - xy = a + 1 \Rightarrow a(x^2 - y^2) + x^2 - xy = a + 1$$

$$\Rightarrow (x - y)\left((a + 1)x + ay\right) = a + 1 \quad \textbf{(7)}$$

$$(6)/(7) \Rightarrow \frac{a^2x - (a + 1)^2y}{(a + 1)x + ay} = \frac{a^2}{a + 1} \Rightarrow y = 0$$

$y = 0$ **in (6)**$\Rightarrow a^2x^3 = a^2 \Rightarrow x = 1$ **and by (5) we find** $z = -a$

Therefore there is only one unique solution $(x, y, z) = (1, 0, -a)$

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

$$\begin{cases} x^2 = yz + 1 \ (A) \\ y^2 = zx + a \ (B) \\ z^2 = xy + a^2 (C) \end{cases}, a > 1$$

$$(A) - (B) \Rightarrow x^2 - y^2 = yz - xz + 1 - a \Rightarrow (x - y)(x + y) = z(y - x) + 1 - a$$

$$\Rightarrow (x - y)(x + y + z) = 1 - a \qquad \textbf{(D)}$$

$$1-a\neq 0\Rightarrow\begin{cases}x+y+z\neq 0\\ x\neq y\end{cases}$$

$$(A)-(C)\Rightarrow x^2-z^2=yz-xy+1-a^2$$

$$\Rightarrow (x-z)(x+z)=y(z-x)+1-a^2$$

$$\Rightarrow (x-z)(x+y+z)=1-a^2=(1-a)(1+a)\quad \textbf{(E)}\ \begin{cases}x+y+z\neq 0\\ x\neq z\end{cases}$$

$$(B)-(C)\Rightarrow y^2-z^2=xz-xy+a-a^2$$

$$\Rightarrow (y-z)(y+z)=x(z-y)+a-a^2$$

$$\Rightarrow (y-z)(x+y+z)=a(1-a)\quad \textbf{(F)}\ \begin{cases}x+y+z\neq 0\\ y\neq 0\end{cases}$$

Replace $(1-a)$ in (E)

$$(x-z)(z+y+z)=(1+a)(x-y)(x+y+z)$$

$$\Rightarrow x-z=(1+a)(x-y)\qquad (x+y+z\neq 0)\Rightarrow z=(a+1)y-ax\quad \textbf{(G)}$$

Replace z in (C): $\left((a+1)y-ax\right)^2=xy+a^2$

$$\Rightarrow a^2y^2+2ay^2+y^2+a^2x^2-2a(a+1)xy-xy-a^2=0$$

$$\Rightarrow a^2y^2+2ay^2+y^2+a^2x^2-2a^2xy-2axy-xy-a^2=0$$

$$\Rightarrow a^2(y^2+x^2-2xy-1)+a(2y^2-2xy)+(y^2-xy)=0$$

$$\Rightarrow\begin{cases}y^2+x^2-2xy-1=0\Rightarrow (y=0)\vee(x=1)\vee(x=-1)\\ 2y^2-2xy=0\Rightarrow (y=0)\vee(x=1)\\ y^2-xy=0\Rightarrow (y=0)\vee(x=1)\end{cases}$$

$y=0$: **(A)** $\Rightarrow x^2=1\Rightarrow (x=1)\vee(x=-1)$

(B) $\Rightarrow xz+a=0\Rightarrow z=-\frac{a}{x}\Rightarrow\begin{cases}(z=-a)\ if\ x=1\\ z=+a\ if\ x=-1\end{cases}$

Finally: 2 solutions: $(x,y,z)=\{(-1,0,a),(1,0,-a)\}$

289. Let $\lambda\in\mathbb{R}$. Solve for real numbers:

$$4^{3x^2-2\lambda x}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}=4^{1+\lambda x}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Mohammed Diai-Rabat-Morocco

$$4^{3x^2-2\lambda x}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}=4^{1+\lambda x}\quad \textbf{(*)}$$

$$\textbf{(*)}\Leftrightarrow 4^{3x^2-2\lambda x}+3\times 4^{2\lambda x-x^2}=4^{1-\lambda x}$$

$$\Leftrightarrow 4^{3x^2-3\lambda x}+3\times 4^{\lambda x-x^2}=4\Leftrightarrow\left(4^{x(x-\lambda)}\right)^3+\frac{3}{4^{x(x-\lambda)}}=4$$

Let be $\alpha=4^{x(x-\lambda)}$ we have the equation: $\alpha^3+\frac{2}{\alpha}=4$ (**)

$$(**)\Leftrightarrow \alpha^4-4\alpha+3=0\Leftrightarrow(\alpha^4-\alpha)-3\alpha+3=0$$

$$\Leftrightarrow \alpha(\alpha-1)(\alpha^2+\alpha+1)-3(\alpha-1)=0$$

$$\Leftrightarrow(\alpha-1)(\alpha^3+\alpha^2+\alpha-3)=0$$

$$\Leftrightarrow(\alpha-1)\left((\alpha^3-1)+(\alpha^2-1)+(\alpha-1)\right)=0$$

$$\Leftrightarrow(\alpha-1)^2(\alpha^2+\alpha+1+\alpha+1+1)=0$$

$$\Leftrightarrow(\alpha-1)^2((\alpha+1)^2+2)=0\Leftrightarrow\alpha=1$$

Therefore $4^{x(x-\lambda)}=1\Rightarrow x(x-\lambda)=0\Rightarrow x=0$ or $x=\lambda$

If S denotes the set of real solutions of (*) then: $S=\{0,\lambda\}$

Solution 2 by Ravi Prakash-New Delhi-India

$$4^{1+\lambda x}=4^{3x^2-2\lambda x}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}+4^{2\lambda x-x^2}\geq$$

$$\geq 4\left[4^{3x^2-2\lambda x+2\lambda x-x^2+2\lambda x-x^2+2\lambda x-x^2}\right]^{\frac{1}{4}}$$

$$[\text{A.M.}\geq\text{G.M.}]\Rightarrow 4^{1+\lambda x}\geq(4)\left(4^{\lambda x}\right)$$

Equality when

$$4^{3x^2-2\lambda x}=4^{2\lambda x-x^2}\Rightarrow 3x^2-2\lambda x=2\lambda x-x^2\Rightarrow x^2=\lambda x\Rightarrow x=0,\lambda$$

290. If $(A,+,\cdot)$ is a ring with $0\neq 1$ and $1+1$ is invertible, then prove that if $a,b\in A$ and $(a+b)^2=a^2+b^2$, $(a+b)^4=a^4+b^4$, then $(ab)^2=0$.

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a^2+b^2=(a+b)^2=(a+b)(a+b)=aa+ba+ab+bb=a^2+ba+ab+b^2;(1)$$

$$ba+ab=0;(2)$$

Also, $a^4+b^4=(a+b)^4=((a+b)^2)^2=(a^2+b^2)^2=$

$$=a^4+b^2a^2+a^2b^2+b^4;(1)\rightarrow$$

$$b^2a^2 + a^2b^2 = 0; (3)$$

But: $b^2a^2 = b(ba)a = b(-ab)a \overset{(2)}{=} -(ba)(ba) = -(ba)^2 = -(-ab)^2; (4)$ *and*

$$a^2b^2 = a(ab)b = a(-ba)b = -(ab)^2; (5)$$

From $(3), (4), (5)$, *we get:* $-(ab)^2 - (ab)^2 = 0 \to -(1+1)(ab)^2 = 0$.

As $(1+1)$ *is invertible,* $-(1+1)$ *is invertible.*

Therefore, $(ab)^2 = 0$.

Solution 2 by Samar Das-India

$(A, +, \cdot)$ *is a ring,* $(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb$, $a^2 + b^2 = (a+b)^2$

$$\to a^2 + b^2 = a^2 + ab + ba + b^2 \to 0 + b^2 = ab + ba + b^2 \to$$

$$b^2 + (-b^2) = ab + ba + b^2 + (-b^2) \to 0 = ab + ba$$

$$\to -ab + 0 = -ab + ab + ba \to -ab = ba$$

Again, $(a+b)^4 = a^4 + b^4 \to ((a+b)^2)^2 = a^4 + b^4 \to$

$$a^2a^2 + a^2b^2 + b^2a^2 + b^2b^2 = a^4 + b^4, (\because (a+b)^2 = a^2 + b^2) \to$$

$$(a^2 + b^2)^2 = a^4 + b^4 \to (a^2 + b^2)(a^2 + b^2) = a^4 + b^4 \to$$

$$a^2a^2 + a^2b^2 + b^2a^2 + b^2b^2 = a^4 + b^4$$

$$-a^4 + a^2a^2 + a^2b^2 + b^2a^2 + b^2b^2 = -a^4 + a^4 + b^4$$

$$0 + a^2b^2 + b^2a^2 + b^2b^2 = 0 + b^4$$

$$a^2b^2 + b^2a^2 + b^2b^2 + (-b^4) = b^4 + (-b^4)$$

$$\to a^2b^2 + b^2a^2 = 0 \to a^2b^2 = -b^2a^2 \text{ and } b^2a^2 = -a^2b^2$$

And now, $(ab)^2 = (ab)(ab) = a(ba)b = a(-a)(bb) = -aabb =$

$$(\because ba = -ab; a(-a) = (-a)a)$$

$$= -a^2b^2 = b^2a^2 = bbaa = b(ba)a = b(-ab)a$$

$$(ab)^2 = b(-a)(ba) = -baba = -(ba)^2 = -((-1)(-1))(ab)^2 = -(ab)^2 \to$$

$$(ab)^2 + (ab)^2 = -(ab)^2 + (ab)^2 = 0 \to (ab)^2 = 0$$

291. $A, B \in M_3(\mathbb{C}), 2021AB = I_3 + 2020BA$. **Find:**

$$\Omega = Tr((AB - BA)^3)$$

Proposed by Marian Ursărescu – Romania

Solution by Ruxandra Daniela Tonila – Romania

$$2021AB = I_3 + 2020BA \Leftrightarrow 2020AB + AB = I_3 + 2020BA$$

$$\Leftrightarrow 2020(AB - BA) = I_3 - AB \Leftrightarrow AB - BA = \frac{1}{2020}(I_3 - AB) \quad \textbf{(1)}$$

$$2021AB = I_3 + 2020BA \Leftrightarrow 2021AB = I_3 - BA + 2021BA$$

$$\Leftrightarrow 2021(AB - BA) = I_3 - BA \Leftrightarrow AB - BA = \frac{1}{2021}(I_3 - BA) \quad \textbf{(2)}$$

Let $C = AB - BA$

From Cayley-Hamilton theorem we have:

$$C^3 - TrC \cdot C^2 + Tr\, C^* \cdot C - \det C \cdot I_3 = O_3$$

$$\Leftrightarrow C^3 = Tr\, C \cdot C^2 - Tr\, C^* \cdot C + \det C \cdot I_3$$

$$\Leftrightarrow Tr\,(C^3) = Tr\,(Tr\, C \cdot C^2 - Tr\, C^* \cdot C + \det C \cdot I_3)$$

$$\Leftrightarrow Tr\,(C^3) = Tr\,\left(Tr\, C \cdot C^2 - \frac{1}{2}(Tr^2C - Tr\, C^2) \cdot C + \det C \cdot I_3\right)$$

$$\Leftrightarrow Tr\,(C^3) = Tr\, C \cdot Tr\, C^2 - \frac{1}{2}Tr\, C \cdot Tr^2C + \frac{1}{2}\, Tr\, C \cdot Tr\, C^2 + 3\det C$$

$$\left.\begin{array}{l}\Leftrightarrow Tr\,(C^3) = Tr\, C\,\left(\frac{3}{2}Tr\, C^2 - \frac{1}{2}Tr^2C\right) + 3\det C \\ Tr\, C = Tr\,(AB - BA) = 0, \forall A, B \in M_3(\mathbb{R})\end{array}\right\} \Rightarrow Tr\,(C^3) = 3\det C \quad \textbf{(3)}$$

$$\textbf{(1), (2)} \Rightarrow \begin{cases} AB - BA = \frac{1}{2020}(I_3 - AB) \\ AB - BA = \frac{1}{2021}(I_3 - BA) \end{cases}$$

$$\Leftrightarrow \begin{cases} \det(AB - BA) = \dfrac{1}{2020^3}\det(I_3 - AB) \\ \det(AB - BA) = \dfrac{1}{2021^3}\det(I_3 - BA) \\ \det(I_3 - AB) = \det(I_3 - BA), \forall A, B \in M_3(\mathbb{R}) \end{cases}$$

$$\Rightarrow \det C = \det(AB - BA) = \det(I_3 - AB) = \det(I_3 - BA) = 0$$

$$\textbf{(3)} \Rightarrow Tr\,((AB - BA)^3) = 3\det(AB - BA) = 0$$

Therefore, $\Omega = 0$.

292. If $A \in M_2(R)$ such that $tr(A) + det(A) = 0$. Prove that:

$$det(A^2 + 3A + 3I_2) + det(A^2 - 3A + 3I_2) \geq 30detA$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer

$$p_A(x)=x^2-(trA)x+detA, \lambda_1+\lambda_2=trA, \lambda_1\lambda_2=detA$$

$$det(A^2+3A+3I_2)=(\lambda_1^2+3\lambda_1+3)(\lambda_2^2+3\lambda_2+3)=$$

$$=\lambda_1^2\lambda_2^2+3\lambda_1^2\lambda_2+3\lambda_1^2+3\lambda_1\lambda_2^2+9\lambda_1\lambda_2+9\lambda_1+3\lambda_2^2+9\lambda_2+9=$$

$$=(detA)^2+3detA\cdot trA+3((trA)^2-2detA)+9trA+9detA+9=$$

$$=(detA)^2-3(detA)^2-3(detA)^2-6detA-9detA+9detA+9=$$

$$=(detA)^2-6detA+9; (1)$$

$$det(A^2-3A+3I_2)=(\lambda_1^2-3\lambda_1+3)(\lambda_2^2-3\lambda_2+3)=$$

$$=\lambda_1^2\lambda_2^2-3\lambda_1^2\lambda_2+3\lambda_1^2-3\lambda_1\lambda_2^2+9\lambda_1\lambda_2-9\lambda_1+3\lambda_2^2-9\lambda_2+9=$$

$$=(detA)^2-3detA\cdot trA+3((trA)^2-2detA)+9trA-9detA+9=$$

$$=(detA)^2+3(detA)^2+3(detA)^2-6detA+9detA+9detA+9=$$

$$=7(detA)^2+12detA+9; (2)$$

From $(1), (2)$ *it follows that:*

$$det(A^2+3A+3I_2)+det(A^2-3A+3I_2)=8(detA)^2+6detA+18=$$

$$=2(4(detA)^2+3detA+9)\geq 30detA$$

Solution 2 by Ruxandra Daniela Tonilă-Romania

Let $p_A=det(A-xI_2)=x^2-(trA)x+detA$ *be the characteristic polynomial of* A.

$$\left.\begin{array}{r}p_A(x)=x^2-(trA)x+detA\\ trA+detA=0\end{array}\right\}\rightarrow p_A(x)=x^2+(detA)x+detA$$

$$det(A^2+3A+3I_2)+det(A^2-3A+3I_2)=$$

$$=det(A^2+3I_2+3A)+det(A^2+3I_2-3A)=$$

$$=2\left(det(A^2+3I_2)+det(3A)\right); (1)$$

$$det(A^2+3I_2)=det(A+i\sqrt{3}I_2)\cdot det(A-i\sqrt{3}I_2)=p_A(-i\sqrt{3})\cdot p_A(i\sqrt{3})$$

$$\left.\begin{array}{r}p_A(-i\sqrt{3})=-3+detA-i\sqrt{3}(detA)\\ p_A(i\sqrt{3})=-3+detA+i\sqrt{3}(detA)\end{array}\right\}\rightarrow$$

$$p_A(-i\sqrt{3})\cdot p_A(i\sqrt{3})=(-3+detA)^2-(detA)^2(i\sqrt{3})^2$$

$$\leftrightarrow det(A^2+3I_2)=4det(A^2)-6detA+9$$

From (1) *it follows that:*

$$det(A^2+3A+3I_2)+det(A^2-3A+3I_2)=$$

$$=2\left(4det(A^2)-6detA+9+det(3A)\right)=6detA+2(4det(A^2)+9)\overset{AM-GM}{\geq}$$

$$\geq 6detA+4\sqrt{4det(A^2)\cdot 9}=6detA+24detA=30detA.$$

Solution 3 by Ravi Prakash-New Delhi-India

$$Let\ A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, detA=ad-bc, trA=a+d=-\alpha$$

$$\because trA+detA=0\ and\ A^2-(trA)A+(detA)I_2=O_2,(H-C)$$

$$\to A^2+\alpha A+\alpha I_2=O_2\to A^2=-\alpha A-\alpha I_2$$

$$Now, A^2+3A+3I_2=(3-\alpha)A+(3-\alpha)I_2=(3-\alpha)^2det(A+I_2)\to$$

$$det(A^2+3A+3I_2)=(3-\alpha)^2[(a+1)(d+1)-bc]=$$

$$=(3-\alpha)^2(\alpha-\alpha+1)=(3-\alpha)^2$$

$$Also, A^2-3A+I_2=-\alpha A-\alpha I_2-3A+3I_2=-(\alpha+3)A+(3-\alpha)I_2$$

$$det(A^2-3A+3I_2)=(\alpha+3)^2\alpha-(\alpha^2-9)\alpha+(3-\alpha)^2=7\alpha^2+12\alpha+9$$

$$Now, det(A^2+3A+3I_2)+det(A^2-3A+3I_2)=$$

$$=7\alpha^2+12\alpha+9+(3-\alpha)^2=8\alpha^2+6\alpha+18=$$

$$=2(2\alpha-3)^2+30\alpha\geq 30\alpha$$

293. If $n\in N, n\geq 3, n$-fixed. Solve for natural numbers:

$$a^n=bc^{n-1}+cb^{n-1}$$

Proposed by Seyran Ibrahimov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$a^n\overset{(*)}{=}bc^{n-1}+cb^{n-1}$$

$$It\ is\ clear\ that\ (0,b,0)\ and\ (0,0,c)\ are\ solutions\ of\ (*).$$

$$Now, we\ assume\ that: a,b,c\in N^*$$

$$Let\ d=gcd(b,c)\to \exists p,q\in N^*\ such\ that\ b=pd\ and\ c=qd\ and\ gcd(p,q)=1$$

$$\to (*)\leftrightarrow a^n=pqd^n(p^{n-2}+q^{n-2})\to d^n/a^n\to d/a$$

$$Let\ m=\frac{a}{d}\in N^*\to (*)\leftrightarrow m^n=pq(p^{n-2}+q^{n-2})$$

$$Since\ gcd(p,q)=1\to gcd(p,q)=gcd(p,p^{n-2}+q^{n-2})=gcd(p^{n-2}+q^{n-2},q)=1$$

$$\to \exists r,s,z\in N^{*}\ such\ that: p=r^n, q=s^n, p^{n-2}+q^{n-2}=z^n=(r^{n-2})^n+(s^{n-2})^n$$

But we know that the equation

$$x^n+y^n=z^n,(n\geq 3), does\ not\ admit\ a\ solution\ on\ N^{*}$$

$$Therefore, S=\{(0,b,0),(0,0,c)|b,c\in N\}.$$

294. Solve for real numbers:

$$\sqrt[7]{x+3}+\sqrt[7]{6-x}=\sqrt[7]{9}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Serlea Kabay-Liberia

$$\text{Let } 9u^7=x+3, u\in R, then\ u\sqrt[7]{9}+\sqrt[7]{6-9u^7+3}=\sqrt[7]{9}$$

$$u\sqrt[7]{9}+\sqrt[7]{9}\cdot\sqrt[7]{1-u^7}=\sqrt[7]{9}\Leftrightarrow u+\sqrt[7]{1-u^7}=1$$

$$\Leftrightarrow \sqrt[7]{1-u^7}=1-u\Leftrightarrow 1-u^7=(1-u)^7\Leftrightarrow$$

$$7u^6-21u^5+35u^4-35u^3+21u^2-7u=0$$

$$7u(u-1)(u^2-u+1)^2=0.\ \ Since\ u^2-u+1\notin R, then\ u\in\{0,1\}$$

$$\text{Therefore, } x\in\{-3,6\}$$

Solution 2 by Surjeet Singhania-India

$$\text{Suppose } f(y)=\sqrt[7]{y}+\sqrt[7]{1-y}, \forall y\in R.$$

$$f(y) \text{ will remain same for } y>1\ and\ y<0\ as\ f(y)=f(1-y).$$

$$f'(y)=\frac{\left(\sqrt[7]{1-y}\right)^6-\left(\sqrt[7]{y}\right)^6}{\left(\sqrt[7]{y-y^2}\right)^6}<0\ for\ y>1\ and\ y\in\left(\frac{1}{2};1\right)$$

$$f'(y)>0\ for\ y\in\left(0,\frac{1}{2}\right)$$

$$\text{Clearly for } y>1, f(1)>f(y)\to f(y)<1. Since\ f(y)=f(1-y)$$

$$f(y)<1,\forall y\in(-\infty,0)\cup(1,\infty)\ and\ for\ 0<y<1, f(y)>1$$

$$\text{Combining all these facts we get } f(y)=1\ for\ y\in\{0,1\}$$

$$f\left(\frac{t}{9}\right)=\sqrt[7]{\frac{t}{9}}+\sqrt[7]{1-\frac{t}{9}}=1\ only\ for\ t\in\{0,9\}$$

$$\text{Therefore, } x\in\{-3,6\}$$

Solution 3 by Christos Tsifakis-Greece

$$\text{For} -3 \le x \le 9 \text{ let } \sqrt[7]{x+3} = a \ge 0 \text{ and } \sqrt[7]{6-x} = b \ge 0 \to \begin{cases} a^7 = x+3 \\ b^7 = 6-x \end{cases}$$

$$\to \begin{cases} a+b = \sqrt[7]{9} \\ a^7+b^7 = 9 \end{cases} \to a^7+b^7 = (a+b)^7 \Leftrightarrow$$

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 = a^7 + b^7 \Leftrightarrow$$

$$7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 = 0 \Leftrightarrow$$

$$ab(7a^5 + 21a^4b + 35a^3b^2 + 35a^2b^3 + 21ab^4 + 7b^5) = 0$$

$$a = 0 \to x = -3$$

$$b = 0 \to x = 6$$

$$7a^5 + 21a^4b + 35a^3b^2 + 35a^2b^3 + 21ab^4 + 7b^5 > 0$$

$$\text{Therefore, } x \in \{-3, 6\}$$

Solution 4 by Fayssal Abdelli-Bejaia-Algerie

$$\sqrt[7]{x+3} + \sqrt[7]{6-x} = \sqrt[7]{9}; (A)$$

$$\text{Let } 9a^7 = 6-x \to \sqrt[7]{6-x} = \sqrt[7]{9}\cdot a \to x + 9a^2 = 6 \to x+3 = 9-9a^7$$

$$\sqrt[7]{x+3} = \sqrt[7]{9(1-a^2)}$$

$$(A) \to \sqrt[7]{9(1-a^7)} + \sqrt[7]{9}\cdot a = \sqrt[7]{9} \to \sqrt[7]{1-a^7} = 1-a \to$$

$$(1-a)(1+a+a^2+\cdots+a^6) = (1-a)^7 \to$$

$$1-a = 0 \to x = -3 \text{ or } 1+a+a^2+\cdots+a^6 = 0; (B)$$

$$(B) \to 1+a+a^2+\cdots+a^6 = a^6 - 6a^5 + 15a^4 - 20a^3 - 15a^2 + 6a + 1$$

$$-7a^5 + 14a^4 - 21a^3 + 14a^2 - 7a = 0 \to a = 0 \to x = 6 \text{ or}$$

$$a^4 - 2a^3 + 3a^2 - 2a + 1 = 0; (C)$$

$$(a \ne 0) \to a^2 - 2a + 3 - \frac{2}{a} + \frac{1}{a^2} = 0$$

$$\left(a^2 + \frac{1}{a^2}\right) - 2\left(a + \frac{1}{a}\right) + 3 = 0$$

$$\text{Let } \overline{x} = a + \frac{1}{a} \to \overline{x}^2 - 2 = a^2 + \frac{1}{a^2}$$

$$(C) \to (\overline{x}^2 - 2) - 2\overline{x} + 3 = 0 \to \overline{x}^2 - 2\overline{x} + 1 = 0 \to (\overline{x} - 1)^2 = 0 \to \overline{x} = 1$$

$$\overline{x} = 1 \to a + \frac{1}{a} = 1 \to a^2 - a + 1 = 0 \ (\Delta < 0) \text{ so } a^2 - a + 1 \ne 0$$

Therefore, $x \in \{-3, 6\}$

295. **Solve for real numbers:**

$$\begin{cases} x, y, z > 0 \\ \dfrac{x^x \cdot y^y \cdot z^z(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})}{\sqrt{x^{y+z} \cdot y^{z+x} \cdot z^{x+y}}} = 1 \\ x + y + z = 1 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Max Wong-Hong Kong

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \overset{AM-GM}{\leq} \frac{x+y}{2} + \frac{y+z}{2} + \frac{z+x}{2} = x + y + z = 1$$

Equality holds iff $x = y = z = \frac{1}{3}$

$$\frac{x^x y^y z^z}{\sqrt{x^{y+z} y^{z+x} z^{x+y}}} = x^{x-\frac{y}{2}-\frac{z}{2}} y^{y-\frac{x}{z}-\frac{z}{2}} z^{z-\frac{x}{2}-\frac{y}{2}} = x^{\frac{3x}{2}-\frac{1}{2}} y^{\frac{3y}{2}-\frac{1}{2}} z^{\frac{3z}{2}-\frac{1}{2}} = e^{\sum_{cyc} \frac{3x-1}{2} \ln x}$$

Define $f(x) = \frac{3x-1}{2} \ln x$, $f: (0, 1] \to \mathbb{R}$

$$f'(x) = \left(\frac{3}{2} - \frac{1}{2x}\right) + \frac{3}{2} \ln x$$

$$f''(x) = \frac{1}{2x^2} + \frac{3}{2x} = \frac{1}{2x^2}(1 + 3x) > 0, \forall x \in (0, 1)$$

$\therefore$ f is convex

$\therefore$ By Jensen's inequality

$$\sum_{cyc} f(x) \leq f\left(\sum_{cyc} x\right) = f(1) = 0$$

As $g(x) = e^x$ is strictly increasing, $g(f(x)) \leq g(f(1)) = g(0) = 1$

$\therefore \frac{x^x y^y z^z}{\sqrt{x^{y+z} y^{z+x} z^{x+y}}} \leq 1$. Equality holds iff $x = y = z = \frac{1}{3}$

$\therefore \frac{x^x y^y z^z(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})}{\sqrt{x^{y+z} y^{z+x} z^{x+y}}} = 1$ iff $x = y = z = \frac{1}{3}$

Otherwise $\frac{x^x y^y z^z(\sqrt{xy} + \sqrt{yz} + \sqrt{zx})}{\sqrt{x^{y+z} y^{z+x} z^{x+y}}} < 1$

Therefore $x = y = z = \frac{1}{3}$

296. $z_1, z_2, z_3 \in C^* -$**different in pairs**, $|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$

$$\prod \left| (z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2| \right| \overset{(*)}{=} \left(\sum |z_1 - z_2| \right)^3 \to AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\Delta ABC \in C(O, R = 1), AB = |z_1 - z_2| = c, BC = |z_2 - z_3| = a, CA = |z_3 - z_1| = b$$

$$\left| (z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2| \right| = |b(z_1 - z_2) + c(z_1 - z_3)| =$$

$$|(a + b + c)z_1 - (az_1 + bz_2 + cz_3)| = 2s\left| z_1 - \frac{az_1 + bz_2 + cz_3}{a + b + c} \right| = 2s.IA$$

$$(*) \leftrightarrow \prod (2s.IA) = (2s)^3 \leftrightarrow \prod \frac{r}{\sin \frac{A}{2}} = 1 \leftrightarrow r^3.\frac{4R}{r} = 1 \leftrightarrow 2r = 1 = R$$

But we know that $R \geq 2r$ *(Euler), with equality only if* ΔABC *is equilateral.*

Therefore,

$$\prod \left| (z_1 - z_2)|z_1 - z_3| + (z_1 - z_3)|z_1 - z_2| \right| = \left(\sum |z_1 - z_2| \right)^3$$

$$\to AB = BC = CA$$

297. If $n \in \mathbb{N}$ **then:**

$$\sum_{k=1}^{n} \frac{k(k-1)}{cos(\pi k)} = \frac{(2n^2 - 1)cos(\pi n) + 1}{4}$$

Proposed by Asmat Qatea-Afghanistan

Solution 1 by Daniel Sitaru-Romania

We will use mathematical induction:

For $n = 1 \to \frac{1(1-1)}{cos(\pi \cdot 1)} = \frac{(2 \cdot 1^2 - 1)cos(\pi \cdot 1) + 1}{4} \leftrightarrow 0 = 0$

$$P(n): \sum_{k=1}^{n} \frac{k(k-1)}{cos(\pi k)} = \frac{(2n^2 - 1)cos(\pi n) + 1}{4}$$

Suppose $P(n)$-true.

$$P(n+1): \sum_{k=1}^{n+1} \frac{k(k-1)}{cos(\pi k)} = \frac{(2(n+1)^2-1)cos(\pi(n+1))+1}{4}$$

$P(n+1)$ –to prove.

$$\sum_{k=1}^{n+1} \frac{k(k-1)}{cos(\pi k)} = \sum_{k=1}^{n} \frac{k(k-1)}{cos(\pi k)} + \frac{(n+1)n}{cos((n+1)\pi)} =$$

$$= \frac{(2n^2-1)cos(\pi n)+1}{4} + \frac{(n+1)n}{cos((n+1)\pi)} = \frac{(2n^2-1)(-1)^n+1}{4} + \frac{(n+1)n}{(-1)^{n+1}} =$$

$$= \frac{(2n^2-1)(-1)^{2n+1}+(-1)^{n+1}+4n(n+1)}{4\cdot(-1)^{n+1}} = \frac{-2n^2+1+4n^2+4n+(-1)^{n+1}}{4\cdot(-1)^{n+1}} =$$

$$= \frac{2n^2+4n+1+(-1)^{n+1}}{4\cdot(-1)^{n+1}} = \frac{(-1)^{n+1}(2n^2+4n+1)+(-1)^{2n+2}}{4\cdot(-1)^{2n+2}} =$$

$$= \frac{(2(n+1)^2-1)cos(\pi(n+1))+1}{4}$$

$$P(n) \to P(n+1)$$

Solution 2 by Ahmed Yackoube Chach-Mauritania

$$f_n(x) = \sum_{k=1}^{n} \frac{x^k}{cos(k\pi)} = \sum_{k=1}^{n} \frac{x^k}{e^{ik\pi}} = \sum_{k=1}^{n} (-x)^k = x \cdot \frac{(-x)^n-1}{x+1}$$

$$f'_n(x) = \frac{n(-x)^n}{x+1} + \frac{(-x)^n}{(x+1)^2} - \frac{1}{(x+1)^2} = \frac{n(x+1)(-x)^n - 1 + (-x)^n}{(x+1)^2}$$

$$f''_n(x) = \frac{\big(n(x+1)((n-1)x+1+n) - 2x\big)(-x)^n}{x(x+1)^3} + \frac{2}{(x+1)^3}$$

Hence,

$$f''_n(1) = \sum_{k=1}^{n} \frac{k(k-1)}{cos(k\pi)} = \frac{(4n^2-2)(-1)^n}{8} + \frac{1}{4} = \frac{(2n^2-1)cos(n\pi)+1}{4}$$

298. **Solve for real numbers:**

$$\begin{cases} x,y,z > 0 \\ \sum_{cyc} x^{2021}\left(\frac{y}{z}+\frac{z}{x}\right) = \sum_{cyc} x^{2020}(y+z) \\ 3^x + 4^y = 5^z \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

$$\sum_{cyc} x^{2021}\left(\frac{y}{z}+\frac{z}{x}\right) = \sum_{cyc} x^{2020}(y+z) \leftrightarrow$$

$$\sum_{cyc} x^{2021}\cdot\frac{y}{z} + \sum_{cyc} x^{2020}\cdot z = \sum_{cyc} x^{2020}\cdot y + \sum_{cyc} x^{2020}\cdot z \leftrightarrow$$

$$\sum_{cyc} x^{2022}\cdot y^2 = \sum_{cyc} x^{2021}\cdot y^2\cdot z$$

$(2021,2,0) \succcurlyeq (2021,2,1)$. Equality if $x=y=z$ and from

$$3^x+4^y=5^z \to x=y=z=2.$$

Solution 2 by Amrit Awasthi-India

$$\sum_{cyc} x^{2021}\left(\frac{y}{z}+\frac{z}{x}\right) = \sum_{cyc} x^{2020}(y+z) \leftrightarrow$$

$$\sum_{cyc} x^{2021}\cdot\frac{y}{z} + \sum_{cyc} x^{2020}\cdot z = \sum_{cyc} x^{2020}\cdot y + \sum_{cyc} x^{2020}\cdot z$$

Or,

$$\sum_{cyc} x^{2020} y\left(\frac{x}{z}-1\right) = 1 \leftrightarrow \sum_{cyc} x^{2020}\left(\frac{y}{z}\right)(x-z) = 0$$

The left hand summand become zero for $x=y=z$.

Now, consider the equation $5^z = 4^y + 3^x$, taking log both sides with base 5

$$z = \frac{\log(3^x+4^y)}{\log 5}$$

The integer solutions of this equation is $(x,y,z)=(0,1,1)$ and using Fermat's last theorem it is concluded that for $x=y=z=a$ have only integer solutions for $a=2$ that is,

$(x,y,z)=(0,1,1)$ or $(2,2,2)$ and putting $x=y=z=a$ in the equations yields

$$a = \frac{\log(3^a+4^a)}{\log 5}$$

Now, we need to check whether other real solutions exist or not.

By considering two equations

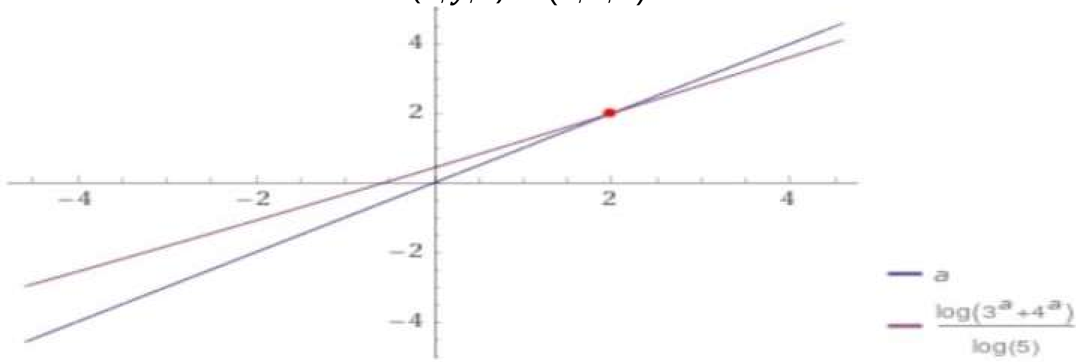
$$Y=a,\ Y=\frac{\log(3^a+4^a)}{\log 5}$$

We can see that both are straight lines.

Now, we know two straight lines either are parallel or coincident or intersect at one point.

But we also have one known solution at value 2 hence lines can not be parallel or coincident. This makes it right to conclude that these lines intersect at $a = 2$ only. Hence the solution set is:

$$(x, y, z) = (2, 2, 2)$$



299. **Prove that, if the real x, y, z, λ satisfy the relation:**

$$\sum x^2 - \sum xy + 3\lambda(\lambda - x + z) = 0,$$

x, y and z form an arithmetic progression.

Proposed by Denisa Lepădatu-Romania

Solution 1 by Fayssal Abdelli-Bejaia-Algerie

Suppose: $y = x + a$ **and** $z = x + 2a, a \in R$

$$\sum x^2 - \sum xy + 3\lambda(\lambda - x + z) = 0; (*)$$

$$\to x^2 + (x + a)^2 + (x + 2a)^2 - x(x + a) - (x + a)(x + 2a) - x(x + 2a) + 3\lambda^2 - 3\lambda x + 3\lambda(x + 2a) = 0$$

$$\to 3a^2 + 6a\lambda + 3\lambda^2 = 0 \to (a + \lambda)^2 = 0 \to a = -\lambda$$

So, $x^2 + y^2 + z^2 - xy - xz - yx + 3\lambda(\lambda - x + z) = 0$

$$\to a = -\lambda, y = x - \lambda, z = x - 2\lambda = y - \lambda$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum x^2 - \sum xy + 3\lambda(\lambda - x + z) = 0; (*)$$

$$(*) \leftrightarrow 3\lambda^2 + 3\lambda(z - x) + \frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] = 0$$

$$x, y, z, \lambda\ exist \leftrightarrow \Delta = 9(z - x)^2 - 4.3.\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \geq 0$$

$$\leftrightarrow 3(z-x)^2-6(x-y)^2-6(y-z)^2\geq 0$$

$$\leftrightarrow [(x-y)+(y-z)]^2-2(x-y)^2-2(y-z)^2\geq 0$$

$$\leftrightarrow -(x-y)^2+2(x-y)(y-z)-(y-z)^2\geq 0 \leftrightarrow -[(x-y)-(y-z)]^2\geq 0$$

$$\rightarrow (x-y)-(y-z)=0\rightarrow x+z=2y.$$

$$Therefore, x, y\ and\ z\ form\ an\ arithmetic\ progression$$

Solution 3 by Bedri Hajrizi-Mitrovica-Kosovo

Let $y=x+a, z=x+b\rightarrow$

$$3x^2+a^2+b^2+2ax+2bx-x(x+a)-x(x+b)-(x+a)(x+b)$$
$$+3\lambda(\lambda-x+x+b)=0$$

$$\rightarrow a^2+b^2-ab+3\lambda^2+3\lambda b=0$$

$$\rightarrow 3\lambda^2+3b\lambda+a^2+b^2-ab=0, \Delta=-3(2a-b)^2\geq 0\Leftrightarrow b=2a$$

So, $y=x+a, z=x+2a\rightarrow x, y, z$ are terms of an arithmetic progression.

Solution 4 by Manole Buican-Romania

$$y^2-(x+z)y+x^2+z^2-xz+3\lambda^2-3\lambda x+3\lambda z=0 \text{ and } \Delta\geq 0$$

$$\Delta=(x+z)^2-4x^2-4z^2+4xz-12\lambda^2+12\lambda x-12\lambda z=$$

$$=-3(x-z)^2-12\lambda\big(\lambda+(x-z)\big)\geq 0, \text{ let } t=x-z\rightarrow$$

$$t^2+4\lambda(\lambda+t)\leq 0\Leftrightarrow (t+2\lambda)^2\leq 0\Leftrightarrow t+2\lambda=0\rightarrow\Delta=0.$$

So, $y_{1,2}=\frac{x+z}{2}\rightarrow x, y, z$- are terms of an arithmetic progression.

Solution 5 by Amrit Awasthi-India

Given, $x^2+y^2+z^2-xy-yz-zx+3\lambda(\lambda-x+z)=0$

$$\frac{1}{2}\left((x-y)^2+(y-z)^2+(z-x)^2\right)+3\lambda(\lambda-x+z)=0;(*)$$

Let $y=\alpha, x=\alpha-k_1, z=\alpha+k_2$ therefore equation $(*)$ becomes

$$\frac{1}{2}\left(k_1^2+k_2^2+(k_1+k_2)^2+3\lambda^2+3\lambda k_1+3\lambda k_2\right)=0$$

$$\rightarrow k_1^2+k_2^2+k_1k_2+3\lambda^2+3\lambda k_1+3\lambda k_2=0$$

Now, the equation so formed is irreducible so let's assume $k_1=k_2=k$, therefore the equation becomes $3k^2+6\lambda k+3\lambda^2=0$, solving for k we get $k=-\lambda$.

→ x, y, z- are terms of an arithmetic progression.

300.

$$\text{If } \psi(x) = \sum_{k=1}^{\infty} \frac{x^k}{d(k)} \text{ then:}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} \psi\left(\frac{e^{\frac{2\pi ik}{n}}}{2}\right) = 1$$

where $|x| < 1$ and $d(n)$ is the number of divisors of n.

Proposed by Angad Singh-Pune-India

Solution by proposer

Let the series expansion of $f(x)$ be expressed as

$$f(x) = \sum_{m=1}^{\infty} a_m x^m \text{, then}$$

$$f\left(xe^{\frac{2\pi ik}{n}}\right) = \sum_{m=1}^{\infty} a_m \left(xe^{\frac{2\pi ik}{n}}\right)^m = \sum_{m=1}^{\infty} a_m x^m e^{\frac{2\pi ikm}{2}}$$

Summing it up from $k = 1$ to $k = n$, we have:

$$\sum_{k=1}^{n} f\left(e^{\frac{2\pi ik}{n}}\right) = \sum_{k=1}^{n} \sum_{m=1}^{\infty} a_m x^m e^{\frac{2\pi ikm}{n}} = \sum_{m=1}^{\infty} a_m x^m \sum_{k=1}^{n} e^{\frac{2\pi ikm}{n}}$$

It can be easily shown that

$$\sum_{k=1}^{n} e^{\frac{2\pi ikm}{n}} = \begin{cases} n, & n|m \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$\sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right) = n \sum_{m=1}^{\infty} a_{mn} x^{mn} \rightarrow \frac{1}{n} \sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right) = \sum_{m=1}^{\infty} a_{mn} x^{mn}$$

Summing up both the sides from $n = 1$ to $n = \infty$, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} x^{mn}$$

Observe that:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}x^{mn} = \sum_{n=1}^{\infty}\left(\sum_{k|n} 1\right) a_n x^n = \sum_{n=1}^{\infty} d(n) a_n x^n$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right) = \sum_{k=1}^{\infty} d(n) a_n x^n$$

Substituting $a_n = \frac{1}{d(n)}$**, if** $|x| < 1$**, we get** $f(x) = \psi(x)$, **thus**

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n} \psi\left(xe^{\frac{2\pi ik}{n}}\right) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$

Finally substituting $x = \frac{1}{2}$ **completes the proof.**

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru