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If $\psi(x)=\sum_{k=1}^{\infty}\frac{x^k}{\sigma(k)}$ then $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\psi\left(\frac{e^{\frac{2\pi ik}{n}}}{2}\right)=1$

$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{xe^{\frac{2\pi ik}{n}}}{1-xe^{\frac{2\pi ik}{n}}}=\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k}$ and $\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{x^m}{\sigma^2(m)}e^{\frac{2\pi ik}{n}}=\psi(x)$

where $|x|<1$ and $\sigma(n)$ is the sum of the divisors of n.

Proposed by Angad Singh-India

Solution by proposer

Let the series expansion of $f(x)$ be expressed as

$$f(x)=\sum_{m=1}^{\infty}a_mx^m \text{ then}$$

$$f\left(xe^{\frac{2\pi ik}{n}}\right)=\sum_{m=1}^{\infty}a_m\left(xe^{\frac{2\pi ik}{n}}\right)^m=\sum_{m=1}^{\infty}a_mx^me^{\frac{2\pi ikm}{n}}$$

Summing it up from $k=1$ to $k=n$, we have

$$\sum_{k=1}^{n}f\left(xe^{\frac{2\pi ik}{n}}\right)=\sum_{k=1}^{n}\sum_{m=1}^{\infty}a_mx^me^{\frac{2\pi ikm}{n}}=\sum_{m=1}^{\infty}a_mx^m\sum_{k=1}^{n}e^{\frac{2\pi ikm}{n}}$$

It can be shown that,

$$\sum_{k=1}^{n}e^{\frac{2\pi ikm}{n}}=\begin{cases}n, & n|m\\0, & otherwise\end{cases} \text{ hence, } \sum_{k=1}^{n}f(xe^{\frac{2\pi ik}{n}})=n\sum_{m=1}^{\infty}a_{mn}x^{mn}$$

Summing up both sides from $n=1$ to $n=\infty$, we have

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} na_{mn}x^{mn}$$

Observe that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}x^{mn}=\sum_{n=1}^{\infty}\left(\sum_{k|n} k\right)a_n x^n=\sum_{n=1}^{\infty}\sigma(n)a_n x^n$$

Therefore,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} f\left(xe^{\frac{2\pi ik}{n}}\right)=\sum_{n=1}^{\infty}\sigma(n)a_n x^n$$

Substituting $a_n=\frac{1}{\sigma(n)}$ and $x=\frac{1}{2}$, if $|x|<1$, we get $f(x)=\psi(x)$, thus

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\psi\left(\frac{e^{\frac{2\pi ik}{n}}}{2}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1$$

This completes the proof of the first equation.

Similarly, if $|x|<1$ and substituting $a_n=1$, we get $f(x)=\frac{x}{1-x}$, thus

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{xe^{\frac{2\pi ik}{n}}}{1-xe^{\frac{2\pi ik}{n}}}=\sum_{n=1}^{\infty}\sigma(n)x^n=\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k}$$

This completes the proof of the second equation.

Finally, if $|x|<1$ and substituting $a_n=\frac{1}{\sigma^2(n)}$, we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\sum_{m=1}^{\infty}\frac{x^m}{\sigma^2(m)}e^{\frac{2\pi ik}{n}}=\sum_{n=1}^{\infty}\frac{x^n}{\sigma(n)}=\psi(x)$$

And this completes the proof of the last equation.