Location of the Kimberling Center X(125) Respect To Orthocentroidal Circle

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Abstract. In this work we give location for Kimberling Center X(125), using barycentric coordinates and $AM-GM$ inequalities.

Keywords. Barycentric Coordinates, Jerabek Hyperbola, Orthocentroidal Circle, $AM-GM$ Inequality

1. Introduction

The orthocentroidal circle $\mathcal{S}_{GH}$ has diameter GH, where G is the centroid and H is the orthocenter of triangle ABC [1]. The orthocentroidal disk of the circle on diameter GH is the interior disc punctured at its center. We use the notation $\mathcal{D}_{GH}$ for the orthocentroidal disk.

The incenter X_1 lies in $\mathcal{D}_{GH}$. Since IGN_a are collinear and $IG : GN_a = 1 : 2$ it follows that Nagel's point X_8 is outside the disk. The symmedian point X_6 lies in the disc $\mathcal{D}_{GH}$. One Brocard point lies in $\mathcal{D}_{GH}$ and the other lies outside $\mathcal{S}_{GH}$, or they both lie simultaneously on $\mathcal{S}_{GH}$ (which happens if and only if the reference triangle is isosceles). X_7, Gergonne's point lies in the orthocentroidal disk $\mathcal{S}_{GH}$. First Fermat's point X_{13} ranges freely over the orthocentroidal disk $\mathcal{D}_{GH}$ and the second Fermat point X_{14} ranges freely over the region external to $\mathcal{S}_{GH}$. The Feuerbach point $Fe = X_{11}$ is always outside the orthocentroidal circle [2].

The Jerabek hyperbola is a circumconic that is the isogonal conjugate of the Euler line [3]. Since it is a circumconic passing through the orthocenter, it is a rectangular hyperbola and has center on the nine-point circle. The Jerabek center is Kimberling center X_{125} [4].

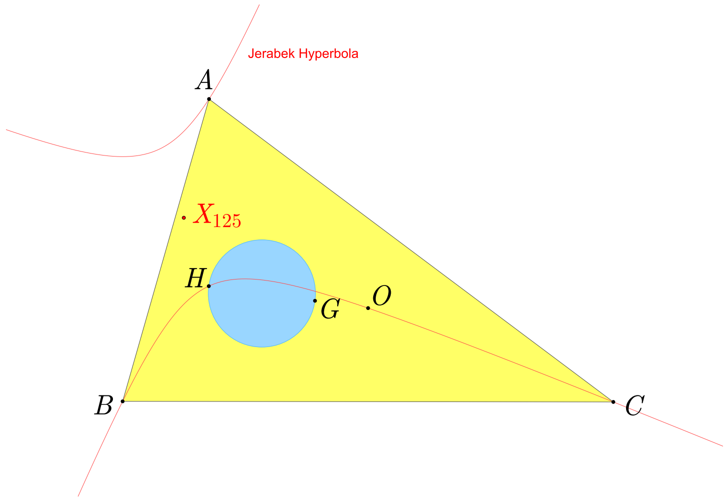

FIGURE 1.

2. THEOREM

Theorem 2.1. *X_{125}, center of Jerabek hyperbola lies outside $\mathcal{S}_{GH}$ (Figure 1).*

Proof. Let $J = X_{381}$ be midpoint of G and H. J has first barycentric coordinate: $(a^4 - 2(b^2-c^2)^2 + a^2(b^2+c^2) ::)$. X_{125} has first barycentric coordinates: $((b^2 - c^2)(a^2(b^2-c^2) - b^4 + c^4) ::)$. We want to show that $JX_{125} > JG$. Using the squared distance $JX_{125}^2 - JG^2$ is equal to:

$$\frac{a^8 - a^6b^2 - a^6c^2 + a^4b^2c^2 - a^2b^6 + a^2b^4c^2 + a^2b^2c^4 - a^2c^6 + b^8 - b^6c^2 - b^2c^6 + c^8}{12(a^6 - a^4b^2 - a^4c^2 - a^2b^4 + 3a^2b^2c^2 - a^2c^4 + b^6 - b^4c^2 - b^2c^4 + c^6)}$$

It's sufficent to show that this expression is positive. Substituting $x = a^2, y = b^2, z = c^2$ in the denominator gives this quantity is positive for all real a, b, c except $a = b = c$. This follows from the well known inequality for non-negative x, y and z that:

$$x^3 + y^3 + z^3 + 3xyz \geq \sum_{sym} x^2y$$

Again substituting $x = a^2, y = b^2, z = c^2$ in the numerator we get $x^4 + y^4 + z^4 + xyz(x+y+z) - (x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3)$. It's positive since $x^4 + y^4 + z^4 \geq xyz(x+y+z)$ gives;

$$2(x^4 + y^4 + z^4) \geq \sum_{sym} x^3y$$

Applying $AM - GM$ to RHS:

$$2(x^4 + y^4 + z^4) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$$
$$x^4 + y^4 + z^4 \geq x^2y^2 + y^2z^2 + z^2x^2$$

which is true by rearrangement ineqaulity. This proves $JX_{125} > JG$. □

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References

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