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NECESSARY AND SUFFICIENT CONDITION SUCH THAT LINE l PASSES THROUGH A CENTER OF A TRIANGLE (LEMMAS)

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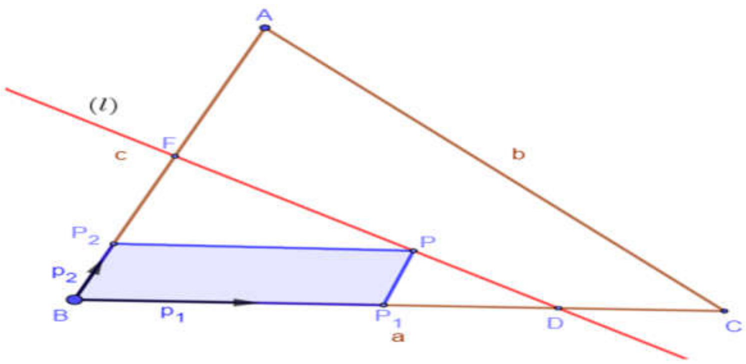
Abstract.

This article is a systematic documentation of the mathematical relationship holds and their proofs, when a line passes through the main centers of a triangle, as a necessary and sufficient condition.Based on a pre-existing relationship (lemma) of a line centroid, I have proved the mathematical relationship, when a line passes through other main centers of a triangle, as it intersects its two sides. More specifically, relationships created for the triangle incenter, excenter, circumcenter, orthocentre,nine-point center, Nagel's point, symmedian point, Spieker's point, Mittenpunkt and Gergonne's point.

At the end of this article, the reader can find two exercises that demonstrate the utility of the proven relationships. I strongly believe that the proposals generated through this study are of a great use, as they could lead us to solutions of similar exercises in a prompt and comprehensive way.

A_0. Line (l) passes through a point P on the plane of the triangle ABC.

Figure-1



Plagiogonal system: $BC \equiv Bx, BA \equiv By$

$$BD = d, BF = f, OP_1 = p_1, OP_1 = p_2$$

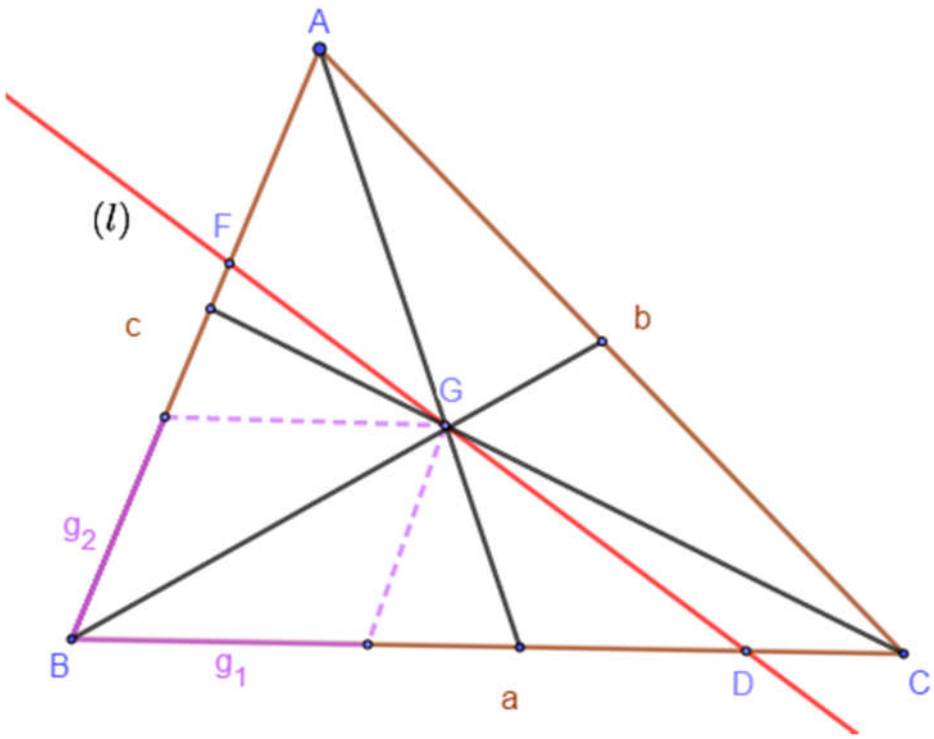
$$B(0,0), D(d,0), C(a,0), F(0,f), A(0,c), P(p_1,p_2)$$

$D, F, P -$ are collinear $\leftrightarrow \begin{vmatrix} 1 & 1 & 1 \\ d & 0 & p_1 \\ 0 & f & p_2 \end{vmatrix} = 0 \leftrightarrow fp_1 + dp_2 = df \leftrightarrow \frac{p_1}{d} + \frac{p_2}{f} = 1; (*)(Lemma)$

Note: The proposal $(*)$ is basic and is used as Lemma to prove the relationship which follow below.

A_1. Line (l) passes through the centroid G of the triangle ABC.

Figure-2



$$g_1 = \frac{a}{3}, g_2 = \frac{c}{3}, BD = d, BF = f$$

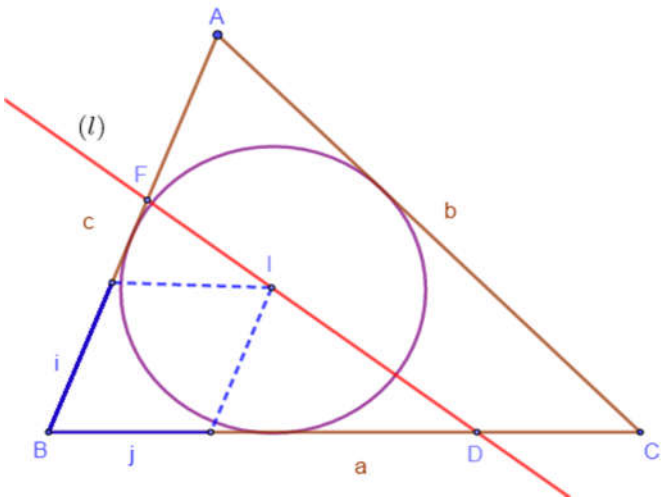
From Lemma$(*)$: $\frac{\frac{a}{3}}{d} + \frac{\frac{c}{3}}{f} = 1 \leftrightarrow \frac{a}{d} + \frac{c}{f} = 1 \leftrightarrow \frac{d+a-d}{d} + \frac{f+c-f}{f} = 3 \leftrightarrow \frac{a-d}{d} + \frac{c-f}{f} = 1$

$D, F, G -$are collinear$\leftrightarrow \frac{BC}{BD} + \frac{BA}{BF} = 3$ or $\frac{CD}{BC} + \frac{AF}{BF} = 1.$

The proposal exists in the mathematical bibliography.

A_2. Line (l) passes through the incenter I of the triangle ABC.

Figure-3



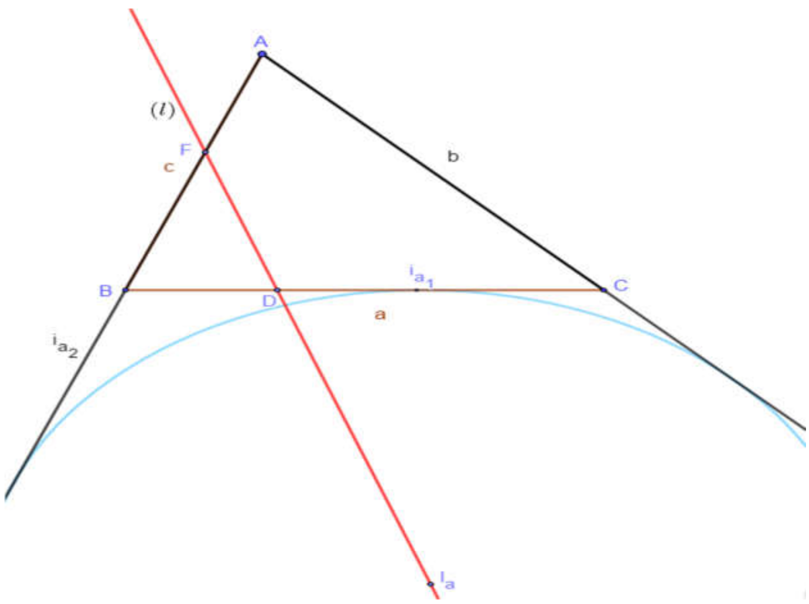
$$i = \frac{ac}{a+b+c}, BD = d, BF = f$$

From Lemma $(*)$: $\frac{i}{d} + \frac{i}{f} = 1 \leftrightarrow \left(\frac{1}{d} - \frac{1}{a}\right) + \left(\frac{1}{f} - \frac{1}{c}\right) = \frac{b}{ac}$

D, F, I –are collinear $\leftrightarrow \left(\frac{1}{BD} - \frac{1}{BC}\right) + \left(\frac{1}{BF} - \frac{1}{BA}\right) = \frac{AC}{BC \cdot BA}$

A_3. Line (l) passes through the excenter I_a of the triangle ABC.

Figure-4



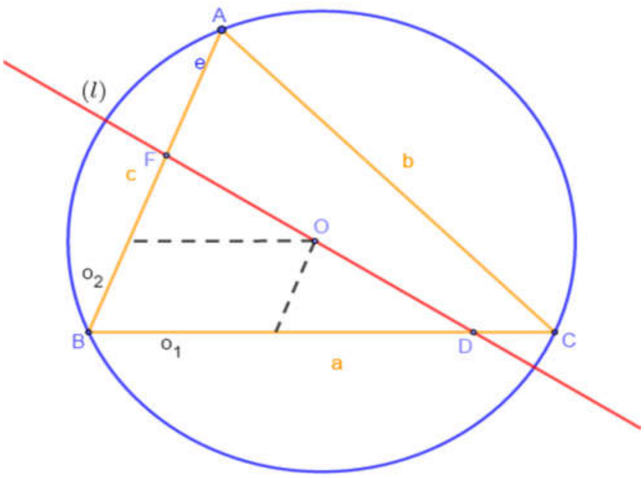
$$i_{a_1}=\frac{ac}{-a+b+c}, i_{a_2}=-\frac{ac}{a+b+c}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{i_{a_1}}{d}+\frac{i_{a_2}}{f}=1\leftrightarrow\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}$

D,F,I –are collinear $\leftrightarrow\left(\frac{1}{BD}-\frac{1}{BC}\right)+\left(-\frac{1}{BF}-\frac{1}{BA}\right)=\frac{AC}{BC\cdot BA}$

A_4. Line (l) passes through the circumcenter O of the triangle ABC.

Figure-5



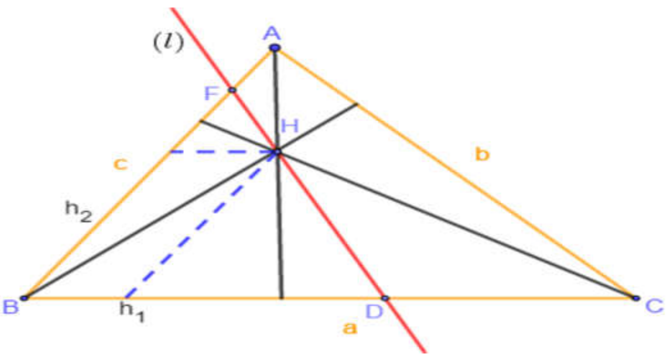
$$o_1=\frac{a-c\cdot\cos B}{2\sin^2 B}, o_2=\frac{c-a\cdot\cos B}{2\sin^2 B}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{o_1}{d}+\frac{o_2}{f}=1\leftrightarrow\left(\frac{a}{d}+\frac{c}{f}\right)-\left(\frac{a}{f}+\frac{c}{d}\right)\cdot\cos B=2\sin^2 B$

D,F,O – are collinear$\leftrightarrow\left(\frac{BC}{BD}+\frac{BA}{BF}\right)-\left(\frac{BC}{BF}+\frac{BA}{BD}\right)\cdot\cos B=2\sin^2 B.$

A_5.Line (l) passes through the orthocenter H of the triangle ABC.

Figure-6



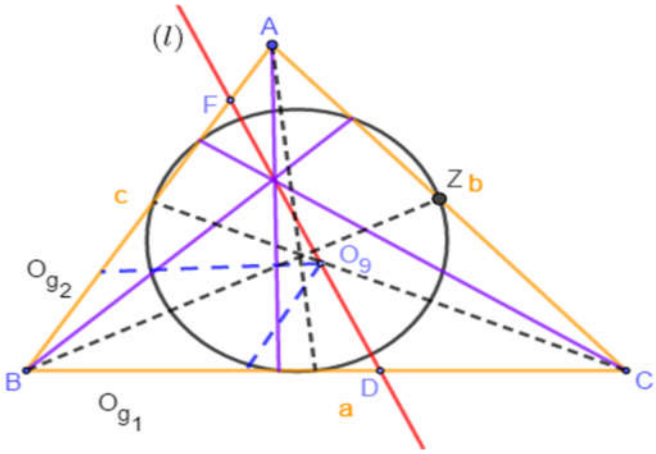
$$h_1 = \frac{(c - a \cdot \cos B)\cos B}{\sin^2 B}, h_2 = \frac{(a - c \cdot \cos B)\cos B}{\sin^2 B}, BD = d, BF = f$$

From Lemma $(*)$: $\frac{h_a}{a} + \frac{h_2}{f} = 1 \leftrightarrow \left(\frac{a}{f} + \frac{c}{d}\right) - \left(\frac{a}{d} + \frac{c}{f}\right)\cos B = \sin B \cdot \tan B$

$$D, F, H - \text{collinear} \leftrightarrow \left(\frac{BC}{BF} + \frac{BA}{BD}\right) - \left(\frac{BC}{BD} + \frac{BA}{BF}\right)\cos B = \sin B \cdot \tan B.$$

A_6. Line (l) passes through the nine point center O_9 of the triangle ABC.

Figure-7



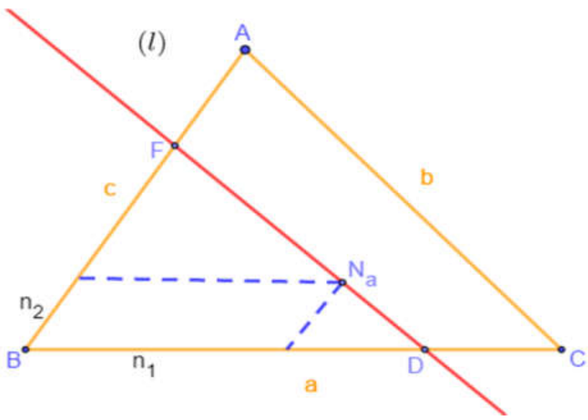
$$O_{9_1} = \frac{c \cdot \cos B - a \cdot \cos 2B}{4\sin^2 B}, O_{9_2} = \frac{a \cdot \cos B - c \cdot \cos 2B}{4\sin^2 B}, BD = d, BF = f$$

From Lemma $(*)$: $\frac{O_{9_1}}{d} + \frac{O_{9_2}}{f} = 1 \leftrightarrow \left(\frac{a}{f} + \frac{c}{f}\right)\cos B - \left(\frac{a}{d} + \frac{c}{f} - 2\right)\cos 2B = 2$

$$D, F, O_9 - \text{collinear} \leftrightarrow \left(\frac{BC}{BF} + \frac{BA}{BD}\right)\cos B - \left(\frac{BC}{BD} + \frac{BA}{BF} - 2\right)\cos 2B = 2.$$

A_7. Line (l) passes through the Nagel's N_a of the triangle ABC.

Figure-8

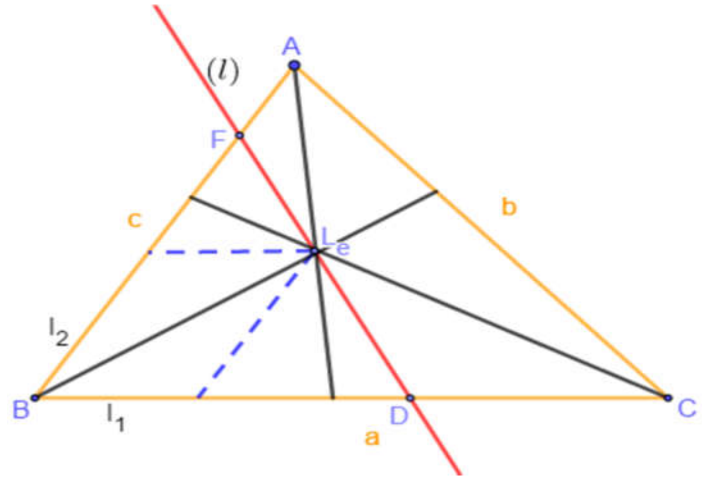


$$n_1=\frac{a+b-c}{a+b+c}\cdot a, n_2=\frac{-a+b+c}{a+b+c}\cdot c, BD=d, BF=f$$

From Lemma $(*)$: $\frac{n_1}{d}+\frac{n_2}{f}=1\leftrightarrow\frac{c}{f}\cdot\frac{s-c}{s}+\frac{BA}{BF}\cdot\frac{s-a}{a}=1.$

A_8. Line (l) passes through the Lemoine's point L_e of the triangle ABC.

Figure-9



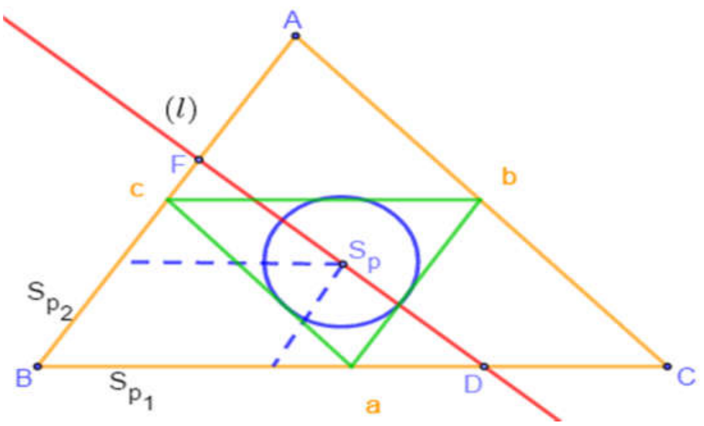
$$l_1=\frac{ac^2}{a^2+b^2+b^2}, l_2=\frac{a^2c}{a^2+b^2+c^2}, BD=d, BF=f$$

From Lemma $(*)$: $\frac{l_1}{d}+\frac{l_2}{f}=1\leftrightarrow\frac{a}{f}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}$

B, D, L_e –collinear$\leftrightarrow\frac{BC}{BF}+\frac{BA}{BD}=\frac{AB^2+BC^2+CA^2}{BA\cdot BC}$

A_9. Line (l) passes through the Spieker's point S_p of the triangle ABC.

Figure-10



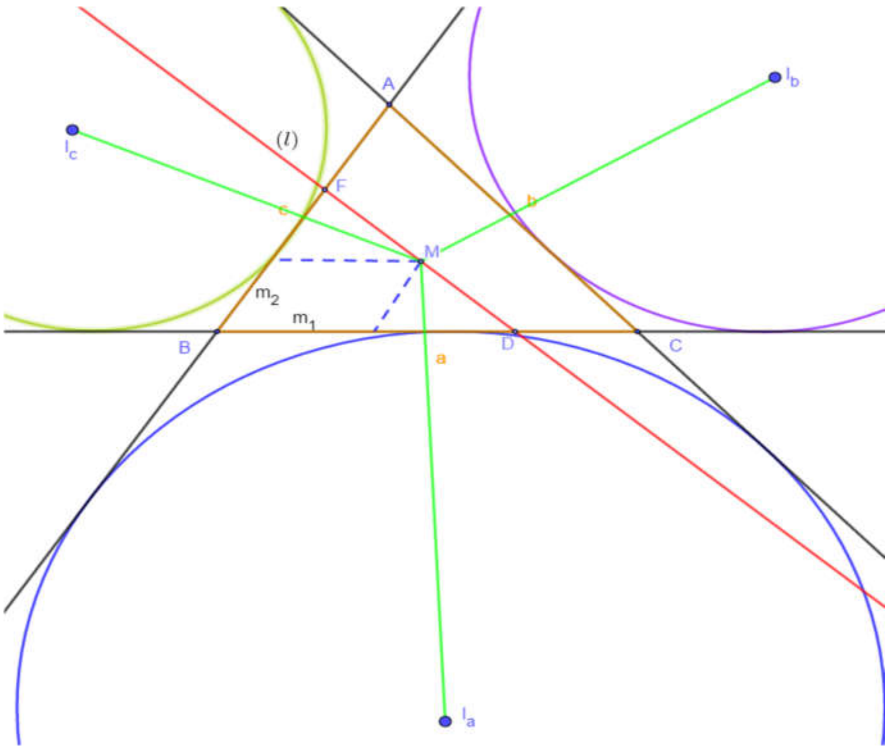
$$S_{p_1} = \frac{a(a+b)}{2(a+b+c)}, S_{p_2} = \frac{c(b+c)}{2(a+b+c)}, BD = d, BF = f$$

From Lemma $(*)$: $\frac{S_{p_1}}{d} + \frac{S_{p_2}}{f} = 1 \leftrightarrow \frac{a}{d}(a+b) + \frac{c}{f}(b+c) = 2(a+b+c)$

D, F, S_p –collinear $\leftrightarrow \frac{BC}{BD}(AC+BC) + \frac{BA}{BF}(AB+AC) = 2(AB+BC+CA)$

A_{10}. Line (l) passes through the Mittenpunkt's point of a triangle ABC.

Figure-11



$$m_1 = \frac{ac(a+b-c)}{(a+b+c)^2 - 2(a^2+b^2+c^2)}, m_2 = \frac{ac(-a+b+c)}{(a+b+c)^2 - 2(a^2+b^2+c^2)}$$

$$BD = d, BF = f$$

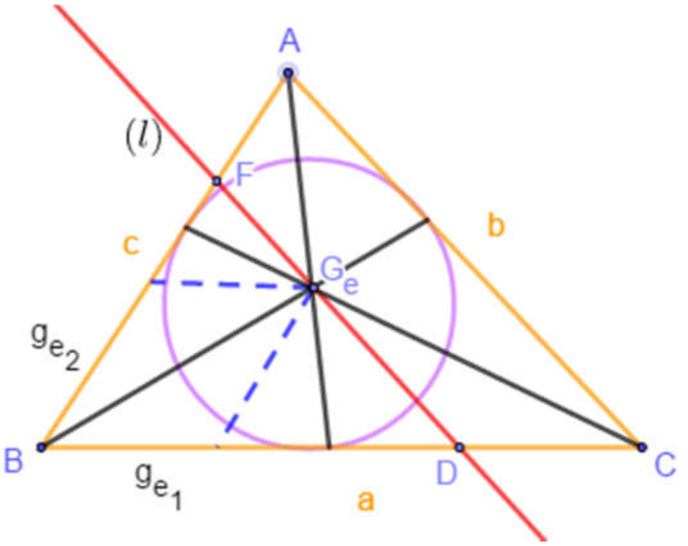
From Lemma $(*)$: $\frac{m_1}{d} + \frac{m_2}{f} = 1 \leftrightarrow \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \frac{1}{2}\left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}\right) = \frac{1}{b}\left(\frac{s-a}{f} + \frac{s-c}{d}\right)$ or

$$\frac{a(s-a) + b(s-b) + c(s-c)}{abc} = \frac{1}{b}\left(\frac{s-a}{f} + \frac{s-c}{d}\right)$$

D, F, M –collinear $\leftrightarrow \frac{a(s-a)+b(s-b)+c(s-c)}{abc} = \frac{1}{b}\left(\frac{s-a}{BF} + \frac{s-c}{BD}\right)$

A_{11}. Line (l) passes therough the Gergonne's point $G_e$of the triangle ABC.

Figure-12



$$g_{e_1}=\frac{a(-a+b+c)(a-b+c)}{2(ab+bc+ca)-a^2-b^2-c^2},g_{e_2}=\frac{c(a+b-c)(a-b+c)}{2(ab+bc+ca)-a^2-b^2-c^2}$$

$$BD=d,BF=f$$

From Lemma $(*)$: $\frac{g_{e_1}}{d}+\frac{g_{e_2}}{f}=1\leftrightarrow\frac{1}{b}\left[\frac{a}{d}(s-a)+\frac{c}{f}(s-c)-\frac{(s-a)(s-c)}{s-b}\right]=1$

D,F,G_e −collinear $\leftrightarrow\frac{1}{b}\left[\frac{a}{BD}(s-a)+\frac{c}{BF}(s-c)-\frac{(s-a)(s-c)}{s-b}\right]=1.$

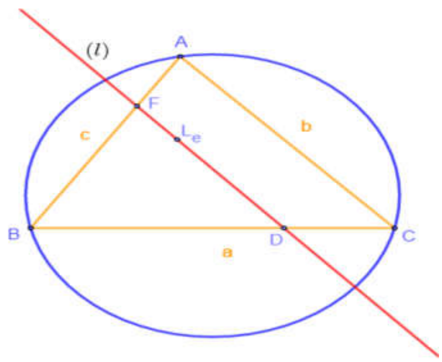
Applications.

Exercise 1. Let L_e −the Lemonine's point of a triangle ABC, line (l) passes through the L_e and intersects the sides BC,BA at the points D,F respectively.

Denote $BD=d,BF=f$ and R −the circumradius of ΔABC. Prove that: $\frac{a}{f}+\frac{c}{d}\geq\frac{b\sqrt{3}}{R}$

Solution.

Figure-13



From (A_8) we have: $\frac{a}{f}+\frac{c}{d}=\frac{a^2+b^2+c^2}{ac}$

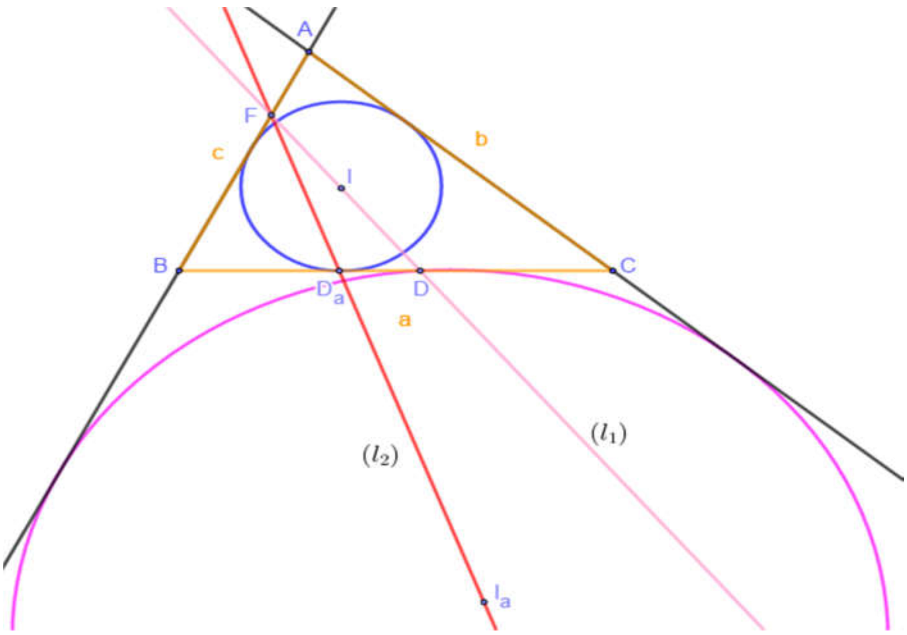
Is $a^2+b^2+c^2\geq 4\sqrt{3}F=4\sqrt{3}\cdot\frac{1}{2}ac\cdot\sin B=\frac{4\sqrt{3}ac}{2}\cdot\frac{b}{2R}$

Hence, $\frac{a}{f}+\frac{c}{d}\geq\frac{b\sqrt{3}}{R}$.

Exercise 2. Line (l_1) passes from incenter I of a triangle ABC and intersects the sides BC, BA at the points D, F respectively. Line (l_2) passes from excenter I_a of the same triangle and intersects the sides BC, BA at the points D_a, F respectively. Denote $BD=d, BD_a=d_a, BF=f$. Prove that: $\frac{1}{d_a}-\frac{1}{d}=2\left(\frac{1}{f}-\frac{1}{c}\right)$.

Solution.

Figure-14



From (A_2) we have: $\left(\frac{1}{d}-\frac{1}{a}\right)+\left(\frac{1}{f}-\frac{1}{c}\right)=\frac{b}{ac}$ and from (A_3) we have:

$$\frac{1}{d_a}-\frac{1}{a}+\left(-\frac{1}{f}+\frac{1}{c}\right)=\frac{b}{ac}$$

Therefore, $\frac{1}{d_a}-\frac{1}{d}=2\left(\frac{1}{f}-\frac{1}{c}\right)$.

Reference:

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