DINCĂ'S REFINEMENT FOR NESBITT'S INEQUALITY

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If $a, b, c > 0$ then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)} \geq \frac{3}{2} \tag{1}$$

Proof.

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a^2}{ab+ac} + \frac{b^2}{bc+ba} + \frac{c^2}{ac+bc} \overset{\text{BERGSTRÖM}}{\geq} \frac{(a+b+c)^2}{2(ab+bc+ca)}$$

Denote $S_1 = a+b+c; S_2 = ab+bc+ca$

$$S_1^2 - 2S_2 = a^2+b^2+c^2 > 0$$

$$S_1^2 - 2S_2 > 0 \Rightarrow S_1^2 > 2S_2 \Rightarrow x = \frac{S_1^2}{S_2} > 2$$

$$\frac{(a+b+c)^2}{2(ab+bc+ca)} \geq \frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)} \Leftrightarrow$$

$$\frac{S_1^2}{2S_2} \geq \frac{3\sqrt{3(S_1^2-2S_2)}}{2S_1}$$

$$S_1^3 \geq 3S_2\sqrt{3(S_1^2-2S_2)}$$

$$S_1^6 \geq 9S_2^2(3(S_1^2-2S_2))$$

$$S_1^6 \geq 27S_1^2S_2^2 - 54S_2^3$$

$$\frac{S_1^6}{S_2^3} - \frac{27S_1^2}{S_2} + 54 \geq 0$$

$$x^3 - 27x + 54 \geq 0$$

$$x > 2 > 0$$

$$x^3 + 54 = x^3 + 27 + 27 \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{x^3 \cdot 27 \cdot 27} = 27x$$

$$x^3 - 27x + 54 \geq 0$$

$$\frac{3\sqrt{3(a^2+b^2+c^2)}}{2(a+b+c)} \geq \frac{3}{2} \Leftrightarrow \sqrt{3(a^2+b^2+c^2)} \geq a+b+c$$

$$3(a^2+b^2+c^2) \geq (a+b+c)^2$$

$$a^2+b^2+c^2 \geq ab+bc+ca$$

$$\frac{1}{2}((a-b)^2+(b-c)^2+(c-a)^2) \geq 0$$

Equality holds in (1) for $a = b = c$.
Observation:
If a, b, c are sides in a triangle then (1) can be written:

$$\frac{a}{2s-a} + \frac{b}{2s-b} + \frac{c}{2s-c} \geq \frac{2s^2}{s^2+r^2+4Rr} \geq \frac{3\sqrt{6(s^2-r^2-4Rr)}}{4s} \geq \frac{3}{2}$$

$\square$

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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