A SIMPLE PROOF FOR WILKER'S INEQUALITY

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WILKER'S INEQUALITY:
If $0 < x < \frac{\pi}{2}$ then:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2$$

Proof.

$$\text{Let be } f : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; f(x) = 2x^2 + x \sin 2x - 4 \sin^2 x$$

$$f'(x) = 4x + \sin 2x + 2x \cos 2x - 4 \sin 2x$$

$$f'(x) = 4x + 2x \cos 2x - 3 \sin 2x$$

$$f'(x) = 2x(2 + \cos 2x) - 3 \sin 2x$$

$$f'(x) = (2 + \cos 2x)\left(2x - \frac{3 \sin 2x}{2 + \cos 2x}\right)$$

$$\operatorname{sgn} f'(x) = \operatorname{sgn}\left(2x - \frac{3 \sin 2x}{2 + \cos 2x}\right) \text{ because } 2 + \cos 2x > 0$$

$$\text{Let be } g : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}; g(x) = 2x - \frac{3 \sin 2x}{2 + \cos 2x}$$

$$g'(x) = 2 - \frac{6 \cos 2x(2 + \cos 2x) - 3 \sin 2x(-2 \sin 2x)}{(2 + \cos 2x)^2}$$

$$g'(x) = \frac{2(2 + \cos 2x)^2 - 6 \cos 2x(2 + \cos 2x) - 6 \sin^2 2x}{(2 + \cos 2x)^2}$$

$$g'(x) = \frac{8 + 8 \cos 2x + 2 \cos^2 2x - 12 \cos 2x - 6 \cos^2 2x - 6 \sin^2 2x}{(2 + \cos 2x)^2}$$

$$g'(x) = \frac{8 - 4 \cos 2x + 2 \cos^2 2x - 6}{(2 + \cos 2x)^2}$$

$$g'(x) = \frac{2(\cos^2 2x - 1)^2}{(2 + \cos 2x)^2} \geq 0$$

$$g \text{ increasing on } \left(0, \frac{\pi}{2}\right) \Rightarrow g(x) > \lim_{\substack{x \to 0 \\ x > 0}} g(x) = 0$$

$$g(x) > 0 \Rightarrow f'(x) = (2 + \cos 2x)g(x) > 0$$

$$f \text{ increasing on } \left(0, \frac{\pi}{2}\right) \Rightarrow f(x) > \lim_{\substack{x \to 0 \\ x > 0}} f(x) = 0$$

$$f(x) > 0; (\forall)x \in \left(0, \frac{\pi}{2}\right)$$

$$2x^2 + x \sin 2x - 4 \sin^2 x > 0$$

$$x^2 + x \sin x \cos x - 2 \sin^2 x > 0$$

$$\left(\frac{x}{\sin x}\right)^2 + x\left(\frac{\cos x}{\sin x}\right) - 2 > 0$$

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2$$

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References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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