A SIMPLE PROOF FOR DOUCET'S INEQUALITY

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DOUCET'S INEQUALITY

In ΔABC the following relationship holds:

$$s\sqrt{3} \leq r + 4R$$

Proof.

$$r_a r_b + r_b r_c + r_c r_a = \frac{F}{s-a} \cdot \frac{F}{s-b} + \frac{F}{s-b} \cdot \frac{F}{s-c} + \frac{F}{s-c} \cdot \frac{F}{s-a} =$$
$$= F^2\Big(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\Big) =$$
$$= s(s-a)(s-b)(s-c)\Big(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\Big) =$$
$$= s(s-c) + s(s-b) + s(s-c) =$$
$$= s(s-c+s-b+s-c) = s^2$$

(1) $$r_a r_b + r_b r_c + r_c r_a = s^2$$

$$r_a + r_b + r_c = \frac{F}{s-a} + \frac{F}{s-b} + \frac{F}{s-c} =$$
$$= F\Big(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\Big) =$$
$$= \frac{F}{(s-a)(s-b)(s-c)}((s-a)(s-c) + (s-c)(s-a) + (s-a)(s-b)) =$$
$$= \frac{Fs}{F^2}(3s^2 - s(b+c+c+a+a+b) + ab + bc + ca) =$$
$$= \frac{s}{F}(3s^2 - s \cdot 4s + ab + bc + ca) =$$
$$= \frac{s}{rs}(-s^2 + s^2 + r^2 + 4Rr) =$$
$$= \frac{1}{r}(r^2 + 4Rr) = r + 4R$$

(2) $$r_a + r_b + r_c = r + 4R$$

If $x, y, z \in \mathbb{R}$ then:

(3) $$3(xy + yz + zx) \leq (x+y+z)^2$$

Replace in $(3) : x = r_a; y = r_b; z = r_c$

$$3(r_a r_b + r_b r_c + r_c r_a) \leq (r_a + r_b + r_c)^2$$

By $(1); (2)$:

$$3s^2 \leq (r+4R)^2$$
$$s\sqrt{3} \leq r + 4R$$

Observation:
By Euler's inequality: $r \leq \frac{R}{2}$

$$s\sqrt{3} \leq r + 4R \leq \frac{R}{2} + 4R \leq \frac{9R}{2}$$

$$2s\sqrt{3} \leq 9R$$

$$s \leq \frac{9R}{2\sqrt{3}} \Rightarrow s \leq \frac{R\sqrt{3}}{2}$$

which is MITRINOVIC'S INEQUALITY

References

[1] Romanian Mathematical Magazine - Interactive Journal, www.ssmrmh.ro

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