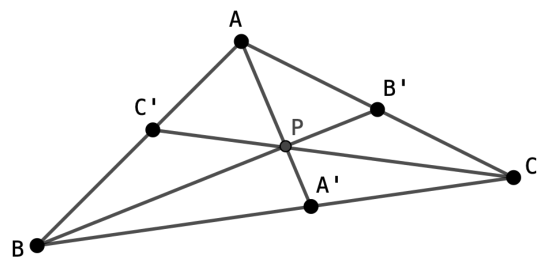
6 AREAS OF 6 FAMOUS PEDAL TRIANGLES

DANIEL SITARU, CLAUDIA NĂNUŢI - ROMANIA

Abstract. In this paper is illustrated the way to find areas of pedal triangles of centroid, incenter, orthocenter (for an acute triangle), symmedian point, Gergonne's point and Nagel's point in a triangle in terms of a, b, c - sides of a given triangle.

Let AA', BB', CC' be concurrent cevians in ΔABC and $\{P\} = AA' \cap BB' \cap CC'$.



By Ceva's theorem:

$$\frac{C'A}{C'B} \cdot \frac{A'B}{A'C} \cdot \frac{B'C}{B'A} = 1$$

Let be $u, v, w > 0$ such that:

$$\frac{C'A}{C'B} = \frac{v}{u}; \frac{A'B}{A'C} = \frac{w}{v}; \frac{B'C}{B'A} = \frac{u}{w}$$

We will find a relationship which can be used to find the distances $AP, BP, CP, A'P, B'P, C'P$ in terms of AA', BB', CC'.

$$\frac{A'B}{A'C} = \frac{w}{v} \Rightarrow \frac{A'B}{A'C + A'B} = \frac{w}{v+w} \Rightarrow \frac{A'B}{a} = \frac{w}{v+w}$$

(1)
$$A'B = \frac{aw}{v+w}$$

$$A'C = BC - A'B = a - \frac{aw}{v+w} = \frac{av}{v+w}$$

(2)
$$A'C = \frac{av}{v+w}$$

By Van Aubel's theorem:

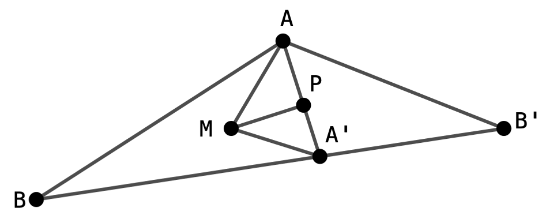
$$\frac{PA}{PA'} = \frac{C'A}{C'B} + \frac{B'A}{B'C} = \frac{v}{u} + \frac{w}{u} = \frac{v+w}{u}$$

$$\frac{PA}{PA + PA'} = \frac{v+w}{v+w+u} \Rightarrow \frac{PA}{AA'} = \frac{v+w}{u+v+w}$$

$$PA = \frac{(v+w)AA'}{u+v+w}$$

$$PA' = AA' - PA = AA' - \frac{(v+w)AA'}{u+v+w} = \frac{u \cdot AA'}{u+v+w}$$

Let be $M \in Int(\Delta ABC)$.



By Stewart's theorem in $\Delta AMA'$:

$$MP^2 \cdot AA' = MA^2 \cdot PA' + MA'^2 \cdot PA - PA \cdot PA \cdot AA'$$

$$MP^2 \cdot AA' = MA^2 \cdot \frac{u \cdot AA'}{u+v+w} + MA'^2 \cdot \frac{(v+w)AA'}{u+v+w} - \frac{u(v+w)}{(u+v+w)^2} \cdot AA'^3$$

(3) $$MP^2 = \frac{u \cdot MA^2}{u+v+w} + \frac{(v+w) \cdot MA'^2}{u+v+w} - \frac{u(v+w)}{(u+v+w)^2} \cdot AA'^2$$

By Stewart's theorem in ΔBMC:

$$MA'^2 \cdot BC = MB^2 \cdot A'C + MC^2 \cdot A'B - A'C \cdot A'B \cdot BC$$

By $(1); (2)$:

$$MA'^2 \cdot a = MB^2 \cdot \frac{av}{v+w} + MC^2 \cdot \frac{aw}{v+w} - \frac{av}{v+w} \cdot \frac{aw}{v+w} \cdot a$$

(4) $$MA'^2 = MB^2 \cdot \frac{v}{v+w} + MC^2 \cdot \frac{w}{v+w} - \frac{a^2vw}{(v+w)^2}$$

By Stewart's theorem in ΔABC:

$$AA'^2 \cdot BC = AB^2 \cdot A'C + AC^2 \cdot A'B - A'B \cdot A'C \cdot BC$$

$$AA'^2 \cdot a = c^2 \cdot \frac{av}{v+w} + b^2 \cdot \frac{aw}{v+w} - \frac{av}{v+w} \cdot \frac{aw}{v+w} \cdot a$$

(5) $$AA'^2 = \frac{c^2v + b^2w}{v+w} - \frac{a^2vw}{(v+w)^2}$$

Replace $(4); (5)$ in (3):

$$MP^2 = \frac{u}{u+v+w} MA^2 + \frac{v+w}{u+v+w}\left(MB^2 \cdot \frac{v}{v+w} + MC^2 \cdot \frac{w}{v+w} - \frac{a^2vw}{(v+w)^2}\right) -$$

$$-\frac{u(v+w)}{(u+v+w)^2}\left(\frac{c^2v+b^2w}{v+w} - \frac{a^2vw}{(v+w)^2}\right)$$

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} - \frac{c^2vu + b^2uw}{(u+v+w)^2} +$$

$$+\frac{a^2vw}{(u+v+w)(v+w)}\left(\frac{u}{u+v+w} - 1\right)$$

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB^2 \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} -$$

$$-\frac{c^2vu + b^2uw}{(u+v+w)^2} + \frac{a^2v \cdot w(u-u-v-w)}{(u+v+w)^2(v+w)}$$

(6)

$$MP^2 = MA^2 \cdot \frac{u}{u+v+w} + MB^2 \cdot \frac{v}{u+v+w} + MC^2 \cdot \frac{w}{u+v+w} - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

Let be $M = O$ - circumcenter. By (6):

$$OP^2 = OA^2 \cdot \frac{u}{u+v+w} + OB^2 \cdot \frac{v}{u+v+w} + OC^2 \cdot \frac{w}{u+v+w} - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

$$OA = OB = OC = R \text{ - circumradii}$$

$$OP^2 = R^2 \cdot \frac{u+v+w}{u+v+w} - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

(7) $$OP^2 = R^2 - \frac{a^2vw + b^2wu + c^2uv}{(u+v+w)^2}$$

1a. For $P = G$ - centroid; $u = v = w = 1; u + v + w = 3$, replaced in (7):

(8) $$OG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}$$

2a. For $P = I$ - incenter; $u = a; v = b; w = c; u + v + w = 2s$, replaced in (7):

$$OI^2 = R^2 - \frac{a^2bc + b^2ca + c^2ab}{4s^2} = R^2 - \frac{abc(a+b+c)}{4s^2} = R^2 - \frac{abc \cdot 2s}{4s^2}$$

$$OI^2 = R^2 - \frac{abc}{2s} = R^2 - \frac{4Rrs}{2s} =$$

(9) $$= R^2 - 2Rr = R(R - 2r)$$

3a. For $P = H$ - orthocenter (ΔABC - acute), $u = \tan A, v = \tan B, w = \tan C$

$$u + v + w = \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Replaced in (7):

$$OH^2 = R^2 - \frac{a^2 \tan B \tan C + b^2 \tan C \tan A + c^2 \tan A \tan B}{(\tan A + \tan B + \tan C)^2}$$

By [1]:

$$\sum_{cyc} a^2 \tan B \tan C = \frac{2F^2}{R^2 \cos A \cos B \cos C}$$

$$OH^2 = R^2 - \frac{1}{(\prod_{cyc} \tan A)^2} \cdot \sum_{cyc} a^2 \tan B \tan C$$

$$OH^2 = R^2 - \frac{1}{(\prod_{cyc} \frac{\sin A}{\cos A})^2} \cdot \frac{2F^2}{R^2 \cdot \prod_{cyc} \cos A}$$

We replace F - area by:

$$F = 2R^2 \prod_{cyc} \sin A$$

$$OH^2 = R^2 - \frac{(\prod_{cyc} \cos A)^2}{(\prod_{cyc} \sin A)^2} \cdot \frac{2 \cdot 4R^4 \cdot (\prod_{cyc} \sin A)^2}{R^2 \cdot \prod_{cyc} \cos A}$$

$$OH^2 = R^2 - 8R^2 \prod_{cyc} \cos A$$

$$OH^2 = R^2\Big(1 - 8\prod_{cyc}\cos A\Big)$$

By [1]:

$$\prod_{cyc}\cos A = \frac{a^2+b^2+c^2-8R^2}{8R^2}$$

$$OH^2 = R^2\Big(1 - 8\cdot\frac{a^2+b^2+c^2-8R^2}{8R^2}\Big)$$

$$OH^2 = R^2 - (a^2+b^2+c^2-8R^2)$$

(10) $$OH^2 = 9R^2 - (a^2+b^2+c^2)$$

4a. For $P = K$ - symmedian point; $u = a^2; v = b^2; w = c^2$.

$$u+v+w = a^2+b^2+c^2 = 2(s^2-r^2-4Rr)$$

Replaced in (7):

$$OK^2 = R^2 - \frac{a^2b^2c^2 + a^2b^2c^2 + a^2b^2c^2}{(a^2+b^2+c^2)^2}$$

(11) $$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}$$

5a. For $P = \Gamma$ - Gergonne's point:

$$u = \frac{1}{s-a}; v = \frac{1}{s-b}; w = \frac{1}{s-c}$$

By [1]: $$\sum_{cyc}\frac{1}{s-a} = \frac{4R+r}{rs}$$

$$u+v+w = \frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c} = \frac{4R+r}{rs}$$

Replaced in (7):

$$O\Gamma^2 = R^2 - \frac{\frac{a^2}{(s-b)(s-c)} + \frac{b^2}{(s-c)(s-a)} + \frac{c^2}{(s-a)(s-b)}}{(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c})^2}$$

$$O\Gamma^2 = R^2 - \Big(\frac{rs}{4R+r}\Big)^2\cdot\frac{1}{(s-a)(s-b)(s-c)}\cdot\sum_{cyc}a^2(s-a)$$

$$\sum_{cyc}a^2(s-a) = s\sum_{cyc}a^2 - \sum_{cyc}a^3 =$$

$$= 2s(s^2-r^2-4Rr) - 2s(s^2-3r^2-6Rr) =$$

$$= 2s(s^2-r^2-4Rr-s^2+3r^2+6Rr) =$$

$$= 2s(2r^2+2Rr) = 4rs(R+r) = 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{r^2s^2}{(4R+r)^2}\cdot\frac{s}{s(s-a)(s-b)(s-c)}\cdot 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{F^2}{(4R+r)^2}\cdot\frac{s}{F^2}\cdot 4F(R+r)$$

$$O\Gamma^2 = R^2 - \frac{4Fs(R+r)}{(4R+r)^2}$$

$$O\Gamma^2 = R^2 - \frac{4rs^2(R+r)}{(4R+r)^2} \tag{12}$$

6a. For $P = N$ - Nagel's point:

$$u = s-a; v = s-b; w = s-c$$

$$u+v+w = 3s-(a+b+c) = 3s-2s = s$$

Replaced in (7):

$$ON^2 = R^2 - \frac{a^2(s-b)(s-c)+b^2(s-c)(s-a)+c^2(s-a)(s-b)}{s^2}$$

$$ON^2 = R^2 - \frac{1}{s^2}\sum_{cyc} a^2(s-b)(s-c)$$

By [1]:

$$\sum_{cyc} a^2(s-b)(s-c) = 4rs^2(R-r)$$

$$ON^2 = R^2 - \frac{1}{s^2}\cdot 4rs^2(R-r)$$

$$ON^2 = R^2 - 4rR + 4r^2 = (R-2r)^2 \tag{13}$$

If ΔDEF is pedal triangle of P then ([2]):

$$[DEF] = \frac{1}{4}(R^2 - OP^2)\cdot\frac{F}{R^2}; F = [ABC]$$

1b. Pedal triangle of centroid:

$$[DEF] = \frac{1}{4R^2}(R^2 - OG^2)\cdot F =$$

$$\overset{(8)}{=} \frac{1}{4R^2}\left(R^2 - R^2 + \frac{a^2+b^2+c^2}{9}\right)\cdot F$$

$$[DEF] = \frac{a^2+b^2+c^2}{36R^2}\cdot F$$

2b. Pedal triangle of incenter:

$$[DEF] = \frac{1}{4R^2}(R^2 - OI^2)\cdot F =$$

$$\overset{(9)}{=} \frac{1}{4R^2}(R^2 - R^2 + 2Rr)F$$

$$[DEF] = \frac{r}{2R}\cdot F$$

3b. Pedal triangle of orthocenter (in acute triangle):

$$[DEF] = \frac{1}{4R^2}(R^2 - OH^2)F =$$

$$\overset{(10)}{=} \frac{1}{4R^2}(R^2 - 9R^2 + a^2 + b^2 + c^2)F =$$

$$= \frac{1}{4R^2}(a^2+b^2+c^2-8R^2)F$$

4b. Pedal triangle of symmedian point:

$$[DEF] = \frac{1}{4R^2}(R^2 - OK^2)F =$$

$$\overset{(11)}{=} \frac{1}{4R^2}\Big(R^2 - R^2 + \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}\Big)F$$

$$[DEF] = \frac{3a^2b^2c^2}{4R^2(a^2+b^2+c^2)^2} \cdot F$$

5b. Pedal triangle of Gergonne's point:

$$[DEF] = \frac{1}{4R^2}(R^2 - O\Gamma^2)F =$$

$$\overset{(12)}{=} \frac{1}{4R^2}\Big(R^2 - R^2 + \frac{4rs^2(R+r)}{(4R+r)^2}\Big)F$$

$$[DEF] = \frac{rs^2(R+r)F}{R^2(4R+r)^2}$$

6b. Pedal triangle of Nagel's point:

$$[DEF] = \frac{1}{4R^2}(R^2 - ON^2)F \overset{(13)}{=} \frac{1}{4R^2}(R^2 - (R-2r)^2)F =$$

$$[DEF] = \frac{1}{4R^2}(R^2 - R^2 + 4Rr + 4r^2)F$$

$$[DEF] = \frac{r(R+r)F}{R^2}$$

References

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com