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UP.359 Find:

$$\Omega=\int_0^\infty \frac{\tan^{-1}x}{x^3+1}dx$$

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Solution 1 by proposer, Solution 2 by Syed Shahabudeen-India

Solution 1 by proposer

$$\Omega(a)=\int_0^\infty \frac{\tan^{-1}ax}{x^3+1}dx, a>0\to \Omega'(a)=\int_0^\infty \frac{x}{(x^3+1)(1+a^2x^2)}dx$$

$$\because b=a^2, \frac{x}{(x+1)(x^2-x+1)(1+bx^2)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}+\frac{Dx+E}{1+bx^2}\to$$

$$A=-\frac{1}{3(b+1)}, B=-\frac{2b-1}{3(b^2-b+1)}, C=\frac{b+1}{3(b^2-b+1)}, D=\frac{b^3}{b^3+1}, E=-\frac{b}{b^3+1}$$

Then, we have:

$$\int \frac{A}{x+1}dx=Alog(x+1)+K$$

$$\int \frac{Bx+C}{x^2-x+1}dx=\frac{B}{2}log(x^2-x+1)+\frac{2C+B}{\sqrt{3}}tan^{-1}\frac{2x-1}{\sqrt{3}}+K$$

$$\int \frac{Dx+E}{bx^2+1}dx = \frac{D}{2b}log(bx^2+1) + \frac{E}{\sqrt{b}}tna^{-1}\sqrt{b}x + K$$

We make the notations:

$$P(x) = \int \frac{x}{(x+1)(x^2-x+1)(1+bx^2)}dx =$$

$$= \int \frac{A}{x+1}dx + \int \frac{Bx+C}{x^2-x+1}dx + \int \frac{Dx+E}{1+bx^2}dx$$

We have:

$$\int_0^\infty \frac{x}{(x^3+1)(1+bx^2)}dx = \lim_{x\to\infty} P(x) - P(0)$$

We can write: $P(x) = Q(x) + R(x)$, where:

$$Q(x) = \frac{2C+B}{\sqrt{3}}tan^{-1}\frac{2x-1}{\sqrt{3}} + \frac{E}{\sqrt{b}}tna^{-1}\sqrt{b}x$$

$$R(x) = Alog(x+1) + \frac{B}{2}log(x^2-x+1) + \frac{D}{2b}log(bx^2+1)$$

Let us denote: $\Delta_1 = \lim_{x\to\infty} Q(x) - Q(0)$, we have:

$$\lim_{x\to\infty} Q(x) = \frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{2} + \frac{E}{\sqrt{b}}\cdot\frac{\pi}{2}; Q(0) = -\frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{6}$$

So, $\Delta_1 = \frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{2} + \frac{E}{\sqrt{b}}\cdot\frac{\pi}{2} + \frac{2C+B}{\sqrt{3}}\cdot\frac{\pi}{6}$, but we have: $2C+B = \frac{b+1}{b^3+1}; E = -\frac{b}{b^3+1} \to$

$$\Delta_1 = \frac{4\pi\sqrt{3} - 9\pi\sqrt{b} + 4\pi\sqrt{3}b}{18(b^3+1)}$$

We make the notation: $\Delta_2(x) = \lim_{x\to\infty} R(x) - R(0)$, where

$$R(x) = Alog(x+1) + \frac{B}{2}log(x^2-x+1) + \frac{D}{2b}log(bx^2+1), R(0) = 0.$$

$$R(x) = Alog\frac{x+1}{x} + \frac{B}{2}log\frac{x^2-x+1}{x^2} + \frac{D}{2b}log\frac{bx^2+1}{bx^2} + \left(A+B+\frac{D}{b}\right)log + \frac{D}{2b}logb$$

But $A+B+\frac{D}{b} = 0$, so $\lim_{x\to\infty} R(x) = \frac{D}{2b}logb = \frac{b^2}{2(b^3+1)}logb \to \Delta_2 = \frac{b^2logb}{2(b^3+1)}$

$$\int_0^\infty \frac{x}{(x^3+1)(1+bx^2)}dx = \Delta_1 + \Delta_2 = \frac{4\pi\sqrt{3} - 9\pi\sqrt{b} + 4\pi\sqrt{3}b + 9b^2logb}{18(b^3+1)}$$

$$\int_0^\infty \frac{x}{(x^3+1)(1+a^2x^2)}dx = \frac{4\pi\sqrt{3} - 9a\pi + 4\pi\sqrt{3}a^2 + 18a^4loga}{18(a^6+1)}$$

$$\Omega = \Omega(1) = \int_0^1 \Omega'(a)\,da, \because \lim_{\substack{a\to 0\\ a>0}} \Omega(a) = 0$$

$$\Omega = \frac{2\pi\sqrt{3}}{9}\int_0^1 \frac{1}{a^6+1}da + \frac{\pi}{2}\int_0^1 \frac{a}{a^6+1}da + \frac{2\pi\sqrt{3}}{9}\int_0^1 \frac{a^2}{a^6+1}da + \int_0^1 \frac{a^4 loga}{a^6+1}da\,;(1)$$

$$\Omega_1 = \int_0^1 \frac{1}{a^6+1}da = \frac{1}{6}\left(\pi + \sqrt{3}log\left(2+\sqrt{3}\right)\right);(2)$$

$$\Omega_2 = \int_0^1 \frac{a}{a^6+1}da = \frac{1}{18}\left(\pi\sqrt{3} + 3log2\right);(3)$$

$$\Omega_3 = \int_0^1 \frac{a^2}{a^6+1}da = \frac{\pi}{12}$$

We calculate the fourth integral:

$$\Omega_4 = \int_0^1 \frac{x^4 logx}{x^6+1}dx$$

For this, we consider the function: $f(x) = \frac{x^4}{x^6+1}$. We develop the function in power series.

We have for $x \in (0,1)$: $f(x) = x^4 - x^{10} + x^{16} - x^{22} + x^{28} - x^{34} + x^{40} - x^{46} + \cdots$

$$\Omega_4 = \int_0^1 f(x)dx = -\frac{1}{5^2} + \frac{1}{11^2} - \frac{1}{17^2} + \frac{1}{23^2} - \frac{1}{29^2} + \frac{1}{35^2} - \frac{1}{41^2} + \cdots$$

$$= -\left(\frac{1}{5^2} + \frac{1}{17^2} + \frac{1}{29^2} + \frac{1}{41^2} + \cdots\right) + \left(\frac{1}{11^2} + \frac{1}{23^2} + \frac{1}{35^2} + \frac{1}{47^2} + \cdots\right)$$

Now, we will use the trigamma function, which is defined by the relationship:

$$\Psi_1(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \to \Omega_4 = \frac{1}{144}\left(-\Psi_1\left(\frac{5}{12}\right) + \Psi_1\left(\frac{11}{12}\right)\right)$$

But, we have the equality: $-\Psi_1\left(\frac{5}{12}\right) + \Psi_1\left(\frac{11}{12}\right) = 4\pi^2\sqrt{3} - 80G$, where G is Catalan's constant. So we have:

$$\Omega_4 = \frac{1}{36}\pi^2\sqrt{3} - \frac{5}{9}G;(5)$$

Replacing the relationships (2),(3),(4),(5) in the relation (1), we obtain:

$$\Omega = \frac{1}{18}\pi^2\sqrt{3} + \frac{1}{9}\pi log\left(2+\sqrt{3}\right) - \frac{1}{12}\pi log2 - \frac{5}{9}G.$$

Solution 2 by Syed Shahabudeen-India

$$\Omega = \int_0^\infty \frac{tan^{-1}x}{x^3+1}dx \overset{x=\frac{1-y}{1+y}}{=} 2\int_{-1}^1 \frac{tan^{-1}\left(\frac{1-y}{1+y}\right)}{6y^2+2}\cdot\frac{dy}{(1+y)^2} =$$

$$= 2\int_{-1}^1 \frac{\left(\frac{\pi}{4}-tan^{-1}x\right)(1+y)}{6y^2+2}dy =$$

$$= \frac{\pi}{2}\int_{-1}^1 \frac{1+y}{6y^2+2}dy - 2\int_{-1}^1 \frac{1+y}{6y^2+2}tan^{-1}y\,dy$$

$$A = \int_{-1}^1 \frac{1+y}{6y^2+2}dy = \frac{1}{3}\int_0^1 \frac{1}{y^2+\frac{1}{3}}dy = \frac{\pi}{3\sqrt{3}}$$

$$B = \int_{-1}^1 \frac{1+y}{6y^2+2}tan^{-1}y\,dy = 2\int_0^1 \frac{ytan^{-1}y}{6y^2+2}dy =$$

$$= 2\left(\frac{\pi}{4}log2 - \frac{1}{12}\int_0^1 \frac{log(3y^2+1)}{y^2+1}dy\right)$$

$$I(a) = \int_0^1 \frac{log(ay^2+1)}{y^2+1}dy$$

$$\frac{dI}{da} = \int_0^1 \frac{y^2}{(ay^2+1)(y^2+1)}dy = \frac{1}{a-1}\int_0^1\left(\frac{1}{y^2+1}-\frac{1}{ay^2+1}\right)dy =$$

$$= \frac{1}{a-1}\left(\frac{\pi}{4}-\frac{tan^{-1}\sqrt{a}}{\sqrt{a}}\right) = \frac{\pi}{4(a-1)} - \frac{tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}} \rightarrow$$

$$I(a) = \frac{\pi}{4}log(a-1) - \int \frac{tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}}da$$

$$\int \frac{tan^{-1}\sqrt{a}}{(a-1)\sqrt{a}}da \overset{m=\sqrt{a}}{=} \frac{\pi}{2}\int\frac{dm}{m^2-1} + 2\int \frac{tan^{-1}\left(\frac{m-1}{m+1}\right)}{m^2-1}dm =$$

$$= \frac{\pi}{4}log\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right) + Ti_2\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right) + C$$

$$I(a) = \frac{\pi}{4}log(a-1) - \frac{\pi}{4}log\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right) - Ti_2\left(\frac{\sqrt{a}-1}{\sqrt{a}+1}\right) + C$$

Ti_2 –Inverse tangent integral. When $a = 0 \rightarrow I = 0$

$$I(0) = -Ti_2(-1) + C = 0 \rightarrow C = -G; (G - Catalan\ ct.)$$

$$I(3)=\frac{\pi}{4}log2+\frac{\pi}{4}log(2+\sqrt{3})-Ti_2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)-G$$

Finally we'll get:

$$\Omega=\frac{\pi^2}{6\sqrt{3}}-\frac{\pi}{6}log2+\frac{\pi}{12}log2+\frac{\pi}{12}log(2+\sqrt{3})-\frac{1}{3}Ti_2\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)-\frac{G}{3}$$

Note by editor:
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