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**UP.356 Find:**

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} \cos\frac{(n-1)k\pi}{n}\cdot \cos^{n-1}\left(\frac{k\pi}{n}\right)}$$

*Proposed by Florică Anastase-Romania*

*Solution by proposer*

$$\because \sum_{k=1}^{n} \cos\frac{(n-1)k\pi}{n}\cdot \cos^{n-1}\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}, \forall n\in\mathbb{N}, n\geq 3$$

$$(1+z)^m = \sum_{l=0}^{m}\binom{m}{l} z^l, m\in\mathbb{N}^*; (1)$$

Let $z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = \overline{1,n} \Rightarrow 1+z = 1+\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} =$

$$= 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right) \Rightarrow$$

$$2^m\cos^m\frac{k\pi}{n}\left(\cos\frac{mk\pi}{n} + i\sin\frac{mk\pi}{n}\right) = \sum_{l=0}^{m}\binom{m}{l}\left(\cos\frac{2lk\pi}{n} + i\sin\frac{2lk\pi}{n}\right) \Rightarrow$$

$$2^m\cos^m\frac{k\pi}{n}\cos\frac{mk\pi}{n} = \sum_{l=0}^{m}\binom{m}{l}\cos\frac{2lk\pi}{n}, k = \overline{1,n} \Rightarrow$$

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\sum_{l=0}^{m}\binom{m}{l}\sum_{k=1}^{n}cos\frac{2ik\pi}{n}=$$

$$=\binom{m}{0}\sum_{k=1}^{n}1+\sum_{i=1}^{n}\binom{m}{i}\sum_{k=1}^{n}cos\frac{2lk\pi}{n};(2)$$

Now,

$$\because\sum_{k=1}^{n}a^{k-1}cos(k\theta)=\frac{a^{n+1}cos(n\theta)-a^ncos(n+1)\theta+cos\theta-a}{a^2-2acos\theta+1};a=1,\theta=\frac{2l\pi}{n}\Rightarrow$$

$$\sum_{k=1}^{n}cos\frac{2lk\pi}{n}=\frac{cos2l\pi-cos\frac{(n+1)2l\pi}{n}+cos\frac{2l\pi}{n}-1}{2-2cos\frac{2l\pi}{n}}=0,\forall l=\overline{1,m};m<n;(3)$$

From $(2),(3)$ it follows that:

$$2^m\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=n\binom{m}{0}\Rightarrow\sum_{k=1}^{n}cos^m\frac{k\pi}{n}cos\frac{mk\pi}{n}=\frac{n}{2^m}$$

For $m=n-1$, it follows that:

$$\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)=\frac{n}{2^{n-1}}$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\sum_{k=1}^{n}cos\frac{(n-1)k\pi}{n}\cdot cos^{n-1}\left(\frac{k\pi}{n}\right)}=\lim_{n\to\infty}\sqrt[n]{\frac{n}{2^{n-1}}}\overset{C-D'A}{=}\lim_{n\to\infty}\frac{n+1}{2^n}\cdot\frac{2^{n-1}}{n}=\frac{1}{2}$$