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SP. 355 Let I_a, I_b, I_c and r_a, r_b, r_c denote the excenters and exradii of the triangle ABC, respectively. Let ρ_a –be the radius of the circle that lies inside and touches internally the excircle opposite A and touches the sides I_aI_b, I_aI_c of triangle $I_aI_bI_c$ externally. Let ρ_b, ρ_c be defined similarly. Prove that:

$$\frac{\rho_a}{r_a}+\frac{\rho_b}{r_b}+\frac{\rho_c}{r_c}\le 1$$

Proposed by Mehmet Şahin-Ankara-Turkey

Solution by proposer

The following equalities can be obtained easily.

$$\sin\left(\frac{B+C}{4}\right)=\frac{\rho_a}{r_a-\rho_a};\ \sin\left(\frac{C+A}{4}\right)=\frac{\rho_b}{r_b-\rho_b};\ \sin\left(\frac{A+B}{4}\right)=\frac{\rho_c}{r_c-\rho_c};(1)$$

$$\frac{\rho_a}{r_a}=\frac{\sin\left(\frac{\pi-A}{4}\right)}{1+\sin\left(\frac{\pi-A}{4}\right)};\ \frac{\rho_b}{r_b}=\frac{\sin\left(\frac{\pi-B}{4}\right)}{1+\sin\left(\frac{\pi-B}{4}\right)};\ \frac{\rho_c}{r_c}=\frac{\sin\left(\frac{\pi-C}{4}\right)}{1+\sin\left(\frac{\pi-C}{4}\right)};(2)$$

Moreover we have:

$$1+\sin\left(\frac{\pi-A}{4}\right)=2\cos^2\left(\frac{\pi+A}{8}\right);\ 1+\sin\left(\frac{\pi-B}{4}\right)=2\cos^2\left(\frac{\pi+B}{8}\right);$$

$$1+\sin\left(\frac{\pi-C}{4}\right)=2\cos^2\left(\frac{\pi-C}{8}\right);(3)$$

Substituing (3) into (2) we get:

$$\frac{\rho_a}{r_a}+\frac{\rho_b}{r_b}+\frac{\rho_c}{r_c}=\frac{2cos^2\left(\frac{\pi+A}{8}\right)-1}{2cos^2\left(\frac{\pi+A}{8}\right)}+\frac{2cos^2\left(\frac{\pi+B}{8}\right)-1}{2cos^2\left(\frac{\pi+B}{8}\right)}+\frac{2cos^2\left(\frac{\pi+C}{8}\right)-1}{2cos^2\left(\frac{\pi+C}{8}\right)}=$$

$$=3-\frac{1}{2}\left[\frac{1}{cos^2\left(\frac{\pi+A}{8}\right)}+\frac{1}{cos^2\left(\frac{\pi+B}{8}\right)+cos^2\left(\frac{\pi+C}{8}\right)}\right];(4)$$

Let $f(x)=\frac{1}{cos^2x}, f''(x)>0\to f$ –convex function, using Jensen's inequality, (4) becomes

$$\frac{1}{cos^2\left(\frac{\pi+A}{8}\right)}+\frac{1}{cos^2\left(\frac{\pi+B}{8}\right)+cos^2\left(\frac{\pi+C}{8}\right)}\ge\frac{3}{cos^2\left(\frac{3\pi+A+B+C}{8\cdot 3}\right)}=\frac{3}{cos^2\frac{\pi}{6}}$$

$$\frac{\rho_a}{r_a}+\frac{\rho_b}{r_b}+\frac{\rho_c}{r_c}\le 3-\frac{1}{2}\cdot 4=1$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.