

Evaluating logarithmic integrals by logarithmic series manipulation

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Abstract

In this article the logarithmic integrals of the following two classes in closed forms

$$\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \left(\frac{a+b}{2} \right) \cdots 1$$
$$\int_0^{\frac{\pi}{2}} \ln \left(p^4 \cos^4 x + \frac{q^4}{16} \sin^4 2x \right) dx = 2\pi \ln \left(\frac{p}{4} \right) + \frac{\pi}{2} \mathcal{A}(p, q) \cdots 2$$

where $\mathcal{A}(p, q) = \ln \left(1 + \sqrt{1 + \frac{q^4}{p^4}} \right) + 2 \ln \left(\sqrt{2} + \sqrt{1 + \sqrt{1 + \frac{q^4}{p^4}}} \right)$ for all $a, b > 0, p > q > 0$ are evaluated using the Maclaurin series of $\log(1+y)$ for $y \in (-1, 1]$.

Introduction

The aforementioned formal integral, [1] is a classical integral that can be found in book, *Integrals, series and products* (see page no 532, section:4.226) and latter integral, [2] is a variant version (due to motivation) of the former integral. The common technique to solve these integrals is Feynman technique however, this paper presents the evaluation of these integrals by series of $\ln(1+y)$ around $y = 0$ that boils down to alternating sum with central binomial coefficients.

Theorems and Proofs

Theorem 1. For all $a, b > 0$, the following integral equality holds.

$$\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln \left(\frac{a+b}{2} \right)$$

Before we develop the proof of **Theorem 1** we need the following lemmas.

Lemma 1.1. For $|x| < \frac{1}{4}$, the generating function of central binomial coefficients $\binom{2n}{n}$ for $n \geq 0$ integer is given by

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

Proof: Consider the function $f(x) = \frac{1}{\sqrt{1+x}}$ for all $x \in (-1, 1]$ and by generalized binomial theorem we write $f(x)$ as

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = \sum_{n=0}^{\infty} \left[\prod_{k=0}^{n-1} \left(-k - \frac{1}{2} \right) \right] \frac{x^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n n!} x^n$$

since $(2n-1)!! = \frac{(2n)!}{2^n n!}$ and replacing x by $-4x$ we have then

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{4^n n!} (-4x)^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$$

Lemma 1.2. Let $n \geq 0$ be integer then the following equality holds.

$$\int_0^{\frac{\pi}{2}} \sin^{2n} u du = \int_0^{\frac{\pi}{2}} \cos^{2n} u du = \frac{\pi}{2 \cdot 4^n} \binom{2n}{n}$$

Proof: Due to *Euler's integral* of the first kind, *Beta function*

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt$$

with the substitution of $t = \sin^2 u$ we obtain

$$B(x, y) = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} u \cos^{2y-1} u du$$

To obtain the desired integral we either set $x = \frac{1}{2}$ or $y = \frac{1}{2}$ and if $y = \frac{1}{2}$ then $x = \frac{2n+1}{2}$ and vice versa.

$$B\left(\frac{2n+1}{2}, \frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \sin^{2n} u du = \int_0^{\frac{\pi}{2}} \cos^{2n} u du$$

since $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and using the relation we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2n} u du = \int_0^{\frac{\pi}{2}} \cos^{2n} u du = \frac{\Gamma\left(n + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{2\Gamma(n+1)} = \frac{\pi}{2 \cdot 4^n} \binom{2n}{n}$$

since we used the relation $\Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n} (2n-1)!! = \frac{(2n)!}{4^n n!} \sqrt{\pi}$.

Proof of Theorem 1

Since $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and thus we write

$$\int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \int_0^{\frac{\pi}{2}} \ln(a^2 \sin^2 x + b^2 \cos^2 x) dx$$

Now for $a > b > 0$ we write $a^2 = (a^2 - b^2) + b^2 = k + b^2$ so we write the integral

$$\mathcal{I}(k, b) = \int_0^{\frac{\pi}{2}} \ln(b^2 + k \cos^2 x) dx = \pi \ln b + \int_0^{\frac{\pi}{2}} \ln\left(1 + \frac{k}{b^2} \cos^2 x\right) dx$$

Now $|\cos^2 x| < 1$ for all $x \in (0, \pi/2)$ and $\frac{a^2 - b^2}{b^2} < 1$ implies $\left|\frac{k}{b^2} \cos^2 x\right| < 1$ and hence

$$\mathcal{J}(k, b) = \int_0^{\frac{\pi}{2}} \ln\left(1 + \frac{k}{b^2} \cos^2 x\right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{k}{b^2}\right)^n \int_0^{\frac{\pi}{2}} \cos^{2n} x dx$$

Now by **Lemma 1.2**, the latter expression boils down to the following infinite sum.

$$\mathcal{J}(k, b) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n n} \left(\frac{k}{b^2}\right) \binom{2n}{n} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{k}{4b^2}\right)^n \binom{2n}{n}$$

To obtain the last series we exploit the **Lemma 1.1**

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \Rightarrow \sum_{n=1}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}} - 1$$

Now by dividing by x and integrating from 0 to y we get

$$\sum_{n=1}^{\infty} \binom{2n}{n} y^n = \int_0^y \frac{1}{x} \left(\frac{1}{\sqrt{1-4x}} - 1\right) dx = -2 \ln\left(\frac{\sqrt{1-4y} + 1}{2}\right)$$

Now setting $y = \frac{-k}{4b^2} = -\frac{a^2 - b^2}{b^2}$

$$\mathcal{J}(k, b) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{k}{4b^2}\right)^n \binom{2n}{n} = \pi \ln\left(\frac{a+b}{2}\right) - \pi \ln b$$

and hence

$$\mathcal{I}(a, b) = \int_0^{\frac{\pi}{2}} \ln(a^2 \cos^2 x + b^2 \sin^2 x) dx = \pi \ln\left(\frac{a+b}{2}\right)$$

and for the case of $b > a$ we note that $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and replacing a by b for k and vice versa the desired same result is obtained.

Theorem 2. For all $p > q > 0$ the following integral equality holds

$$\int_0^{\frac{\pi}{2}} \ln \left(p^4 \cos^4 x + \frac{q^4}{16} \sin^4 2x \right) dx = 2\pi \ln \left(\frac{p}{4} \right) + \frac{\mathcal{A}(p, q)}{2} \pi$$

where $\mathcal{A}(p, q) = \ln \left(1 + \sqrt{1 + \frac{q^4}{p^4}} \right) + 2 \ln \left(\sqrt{2} + \left(1 + \sqrt{1 + \frac{q^4}{p^4}} \right)^{1/2} \right)$

To work with **Theorem 2** we need the following lemma.

Lemma 2.1 For $|x| < \frac{1}{16}$, the generating function for the coefficients $\binom{4n}{2n}$ for all $n \geq 0$ is given by

$$\sum_{n=0}^{\infty} \binom{4n}{2n} (-x)^n = \frac{1}{\sqrt{2}} \sqrt{\frac{1 + \sqrt{1 + 16x}}{1 + 16x}}$$

Proof: Let the function $\mathcal{F}(x) = \frac{1}{\sqrt{1 - 4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$ by **Lemma 1.1** and it is easy to see that

$$\sum_{n=0}^{\infty} \binom{4n}{2n} (-x^2)^n = \Re \left(\sum_{n=0}^{\infty} \binom{2n}{n} (ix)^n \right) = \Re \left(\frac{1}{\sqrt{1 - 4ix}} \right)$$

Now to evaluate the real part, let $\frac{1}{\sqrt{1 - 4ix}} = re^{i\theta}$ and $\Re(\mathcal{F}(ix)) = r \cos \theta$. Here $\cos 2\theta = r^2$ and $\sin 2\theta = 4r^2 x$ and hence $\theta = \frac{1}{2} \arctan 4x$ and $r = \frac{1}{\sqrt[4]{1 + 16x^2}}$. Therefore,

$$\Re(\mathcal{F}(ix)) = \frac{\cos \left(\frac{1}{2} \arctan 4x \right)}{\sqrt{1 + 16x^2}} = \frac{1}{\sqrt{2}} \frac{\sqrt{1 + \cos \arctan 4x}}{\sqrt[4]{1 + 16x^2}} = \frac{1}{\sqrt{2}} \sqrt{\frac{1 + \sqrt{1 + 16x^2}}{1 + 16x^2}}$$

we used $\cos(\arctan x) = \frac{1}{\sqrt{1 + x^2}}$ and on replacing x by $4x$ and simplification gives us the equality right hand side. Moreover, replacing x^2 by x yields the desired result

$$\sum_{n=0}^{\infty} \binom{4n}{2n} (-x)^n = \frac{1}{\sqrt{2}} \sqrt{\frac{1 + \sqrt{1 + 16x}}{1 + 6x}}$$

which completes the proof.

Remark:

$$\sum_{n=0}^{\infty} \binom{4n}{2n} x^{2n} = \frac{\mathcal{F}(x) + \mathcal{F}(-x)}{2} = \frac{1}{2} \left(\frac{1}{\sqrt{1 - 4x}} + \frac{1}{\sqrt{1 + 4x}} \right)$$

Lemma 2.2 We show that

$$\int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$$

Using the integral property $\int_a^b f(x) dx = \int_a^b f(b+a-x) dx$. We directly get the result $\int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \ln(\sin x) dx$. For all $0 \leq x < \frac{\pi}{2}$ we use the *Fourier series* of $\ln(\cos x)$ we have

$$\int_0^{\frac{\pi}{2}} \ln(\cos x) dx = \int_0^{\frac{\pi}{2}} \left(-\ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cos(2kx) \right) dx = -\frac{\pi}{2} \ln 2 - 0 = -\frac{\pi}{2} \ln 2$$

Since

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \int_0^{\frac{\pi}{2}} \cos(2kx) dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(2k\pi) = 0$$

as $\sin(2\pi k) = 0$ for all k (integers).

Proof of Theorem 2

Since

$$\int_0^{\frac{\pi}{2}} \ln \left(p^4 \cos^4 x + \frac{q^4}{16} \sin^4 2x \right) dx = \int_0^{\frac{\pi}{2}} \ln(\cos^4 x) dx + \int_0^{\frac{\pi}{2}} \ln(p^4 + q^4 \sin^4 x) dx$$

Since the formal integral $\int_0^{\frac{\pi}{2}} \ln(\cos^4 x) dx = 4 \int_0^{\frac{\pi}{2}} \ln(\cos x) dx = -2\pi \ln 2$ by

Lemma 2.2. Note that

$$\int_0^{\frac{\pi}{2}} \ln(p^4 + q^4 \sin^4 x) dx = 2\pi \ln p + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{q^4}{p^4} \right)^n \int_0^{\frac{\pi}{2}} \sin^{4n} x dx$$

By the **Lemma 1.2** we have

$$\mathcal{S}(p, q) = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{q^4}{16p^4} \right)^n \binom{4n}{2n}$$

We now evaluate last sum by the use of the **Lemma 2.1** by dividing x and integrating from 0 to z .

$$\sum_{n=1}^{\infty} \frac{(-z)^n}{n} \binom{4n}{2n} = \int_0^z \frac{1}{x} \left(\sqrt{\frac{1 + \sqrt{1 + 16x}}{1 + 16x}} - \sqrt{2} \right) \frac{dx}{\sqrt{2}}$$

We evaluate the indefinite integral of latter result by making substitution $1 + 16x = y$ gives us

$$\frac{1}{\sqrt{2}} \int \left(-\frac{\sqrt{1+\sqrt{y}} + \sqrt{2y}}{(1-y)\sqrt{y}} \right) dy \stackrel{u=\sqrt{y}}{=} -\frac{2}{\sqrt{2}} \int \frac{\sqrt{1+u} - \sqrt{2}u}{1-u^2}$$

further substitute $\sqrt{1+u} = w$ gives us

$$-\frac{4}{\sqrt{2}} \int \frac{w^2 - \sqrt{2}w^3 + \sqrt{2}w}{(w^2-1)^2-1} dw = \frac{4}{\sqrt{2}} \int \frac{(\sqrt{2}w+1)(w-\sqrt{2})}{w(w+\sqrt{2})(w-\sqrt{2})} = \frac{4}{\sqrt{2}} \int \frac{\sqrt{2}w+1}{w(w+\sqrt{2})}$$

and last integral on RHS $\int \frac{\sqrt{2}w+1}{w(w+\sqrt{2})} = \sqrt{2} \ln \left(\frac{w}{\sqrt{2}} + w \right)$ and making undo of each substitution made with simplification we yield

$$\int \frac{1}{x} \left(\sqrt{\frac{1+\sqrt{1+16x}}{1+16x}} - \sqrt{2} \right) \frac{dx}{\sqrt{2}} = -2 \ln \left(\sqrt{1+\sqrt{1+16x}} + \frac{\sqrt{1+16x}+1}{\sqrt{2}} \right) + C$$

and applying the limits we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \binom{4n}{2n} z^n = 2 \ln \underbrace{\left(\sqrt{1+\sqrt{1+16z}} + \frac{\sqrt{1+16z}+1}{\sqrt{2}} \right)}_M - 3 \ln 2$$

Also

$$M = \ln(1 + \sqrt{1+16z}) + 2 \ln \left(\sqrt{2} + \sqrt{1 + \sqrt{1+16x}} \right) - \ln 2$$

for $z = \frac{q^4}{16p^4}$ we obtain the closed form of $\mathcal{S}(p, q) =$

$$\frac{\pi}{2} \underbrace{\left(\ln \left(1 + \sqrt{1 + \frac{q^4}{p^4}} \right) + 2 \ln \left(\sqrt{2} + \sqrt{1 + \sqrt{1 + \frac{q^4}{p^4}}} \right) \right)}_{\mathcal{A}(p,q)} - 2\pi \ln 2$$

Combining the result we obtain the result

$$\int_0^{\frac{\pi}{2}} \ln \left(p^4 \cos^4 x + \frac{q^4}{16} \sin^4 x \right) dx = 2\pi \ln \left(\frac{p}{4} \right) + \frac{\mathcal{A}(p, q)}{2} \pi$$

as required which completes the proof.

Remarkable Result from study

As we proved hat

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \binom{4n}{2n} z^n = M - 3 \ln 2$$

It is interesting to note that sum on left hand attains the hypergeometric expression, namely

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} \binom{4n}{2n} z^n = \frac{3}{8} {}_4F_3 \left(1, 1, \frac{5}{4}, \frac{7}{4}; \frac{3}{2}, 2, 2; -z \right) z$$

In other words $\frac{3}{8} {}_4F_3 \left(1, 1, \frac{5}{4}, \frac{7}{4}; \frac{3}{2}, 2, 2; -z \right) z$

$$= \ln(1 + \sqrt{1 + 16z}) + 2 \ln \left(\sqrt{2} + \sqrt{1 + \sqrt{1 + 16z}} \right) - 3 \ln 2$$

References

[1] I.S Gradshteyn and I.M Ryznik, *Tables of integrals, series and products*, 7th edition, edited by Alan Jeffrey and Daniel Zwillinger.

[2] <https://mathsworld.wolfram.com/CentralBinomialCoefficient.html>.

[3] <https://en.m.wikipedia.org/wiki/Wallis>