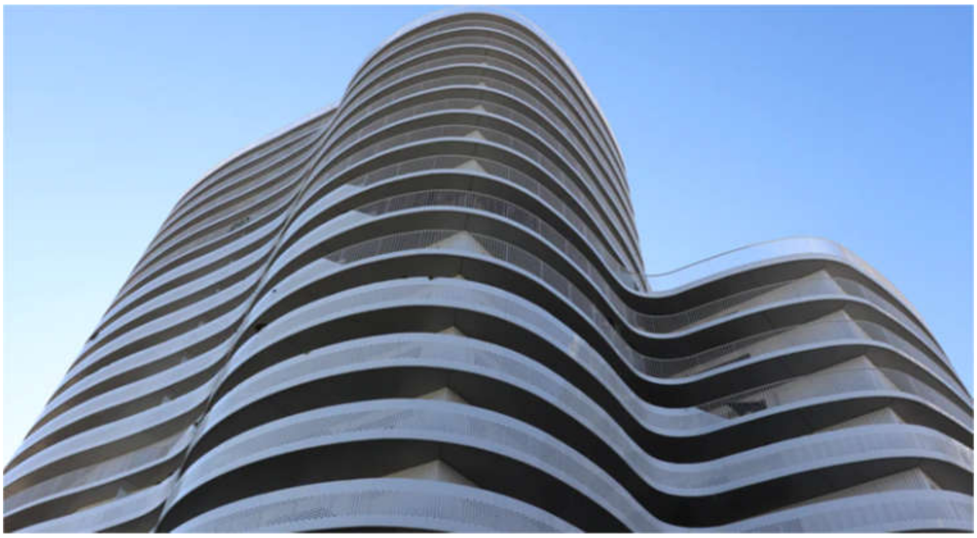
R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro



If $(a_n)_{n\geq 0}$ is sequence of real numbers such that $a_0 = a_1 = 0$ and

$a_{n+2} - 2a_{n+1} + a_n = n + 1$ then find:

$$\Omega = \lim_{n\to\infty} \sum_{k=1}^{n} \log\left(1 - \frac{k^2(k^2-1)}{18a_k a_{k+1}}\right)$$

Proposed by Florică Anastase-Romania

Solution 1 by George Florin Şerban-Romania, Solution 2 by Adrian Popa-Romania, Solution 3 by Ravi Prakash-New Delhi-India, Solution 4 by proposer

Solution 1 by George Florin Şerban-Romania

$$(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) = n + 1. \text{ Let } x_n = a_{n+1} - a_n \to$$

$$x_{n+1} - x_n = n + 1,$$

$$x_1 - x_0 = 1, x_2 - x_1 = 2, \dots, x_n - x_{n-1} = n \to$$

$$x_n - x_0 = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$a_1 - a_0 = 0, a_2 - a_1 = 1, \dots, a_{n+1} - a_n = \frac{n(n+1)}{2} \to$$

$$a_{n+1} - a_0 = \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{4}$$

$$a_n = \frac{(n-1)n(n+1)}{6}$$

$$18a_ka_{k+1}=18\cdot\frac{(k-1)k(k+1)}{6}\cdot\frac{k(k+1)(k+2)}{6}=\frac{(k-1)k^2(k+1)^2(k+2)}{2}$$

$$1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}=1-\frac{2k^2(k-1)(k+1)}{(k-1)k^2(k+1)^2(k+2)}=1-\frac{2}{(k+1)(k+2)}$$

$$=\frac{k(k+3)}{(k+1)(k+2)}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(\frac{k(k+3)}{(k+1)(k+2)}\right)=$$

$$=\lim_{n\to\infty}log\left(\prod_{k=1}^{n}\frac{k(k+3)}{(k+1)(k+2)}\right)=log\left(\lim_{n\to\infty}\frac{n+3}{3(n+1)}\right)=log\left(\frac{1}{3}\right)$$

Solution 2 by Adrian Popa-Romania

$$\begin{cases}a_2-2a_1+a_0=1\\a_3-2a_2+a_1=2\\a_4-2a_3+a_2=2\\\dots\dots\dots\\a_{n+2}-2a_{n+1}+a_n=n+1\end{cases}$$

$$\to a_{n+2}-a_{n+1}=1+2+3+\dots+(n+1)=\frac{(n+1)(n+2)}{2}$$

$$\begin{cases}a_2-a_1=\frac{1\cdot 2}{2}\\a_3-a_2=\frac{2\cdot 3}{2}\\\dots\dots\dots\\a_n-a_{n-1}=\frac{(n-1)n}{2}\end{cases}$$

$$\to a_n=\frac{1\cdot 2+2\cdot 3+\dots+(n-1)n}{2}=\frac{1}{2}\sum_{k=1}^{n}k(k-1)=\frac{1}{2}\sum_{k=1}^{n}(k^2-k)=$$

$$=\frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right)=\frac{n(n-1)(n+1)}{6}$$

$$a_k\cdot a_{k+1}=\frac{k(k-1)(k+1)}{6}\cdot\frac{(k+1)(k+2)k}{6}=\frac{k^2(k+1)^2(k-1)(k+2)}{36}$$

$$\frac{k^2(k^2-1)}{18a_ka_{k+1}}=\frac{2}{(k+1)(k+2)}$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{2}{(k+1)(k+2)}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(\frac{k(k+3)}{(k+1)(k+2)}\right)=\lim_{n\to\infty}log\left(\prod_{k=1}^{n}\frac{k}{k+1}\cdot\frac{k+3}{k+2}\right)=$$

$$=\lim_{n\to\infty}log\left(\frac{n+3}{3(n+1)}\right)=log\left(\frac{1}{3}\right)=-log3$$

Solution 3 by Ravi Prakash-New Delhi-India

Let $a_{n+1}-a_n=b_n,\forall n\geq 0\to b_0=0$. Also, $b_{n+1}-b_n=n+1$

$$\because (a_{n+2}-a_{n+1})-(a_{n+1}-a_n)=n+1$$

$$\to\sum_{k=0}^{n}(b_{k+1}-b_k)=\sum_{k=0}^{n}(k+1)\to b_{n+1}-b_0=\frac{1}{2}(n+1)(n+2)$$

$$\to b_{n+1}=\binom{n+2}{2}\to b_n=\binom{n+1}{2},\forall n\geq 0$$

Thus, $a_{n+1}-a_n=\binom{n+1}{2},\forall n\geq 0\to$

$$\sum_{k=0}^{n}(a_{k+1}-a_k)=\sum_{k=0}^{n}\binom{k+1}{2}\to a_{n+1}-a_0=\binom{2}{2}+\binom{3}{3}+\cdots+\binom{n+1}{2}$$

$$a_{n+1}-a_0=\binom{n+2}{3}\to a_n=\binom{n+1}{3},\forall n\geq 0$$

Thus,

$$1-\frac{k^2(k^2-2)}{18\binom{k+1}{3}\binom{k+2}{3}}=1-\frac{2k^2(k^2-1)}{(k+1)(k+1)^2k^2(k-1)}=1-\frac{2}{(k+2)(k+1)}$$

$$=\frac{k(k+3)}{(k+1)(k+2)}$$

$$\sum_{k=1}^{n}log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)=\sum_{k=1}^{n}\left(logk+log(k+3)-log(k+1)-log(k+2)\right)=$$

$$=log1+log(n+3)-log3-log(n+1)=log\left(\frac{1+\frac{3}{n}}{1+\frac{1}{n}}\right)-log3$$

Therefore,

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)=-log3$$

Solution 4 by proposer

$$\begin{cases}a_{n+2}-2a_{n+1}+a_n=n+1\\a_{n+3}-2a_{n+2}+a_{n+1}=n+2\end{cases}\Rightarrow a_{n+3}-2a_{n+2}+3a_{n+1}-a_n=1,\forall n\in\mathbb{N}$$

$$a_{n+4}-4a_{n+3}+6a_{n+2}-4a_{n+1}+a_n=1$$

$$r^4-4r^3+6r^2-4r+1=0\Rightarrow r_i=0,i=\overline{1,4}\Rightarrow$$

$$a_n=An^3+Bn^2+Cn+D,\forall n\in\mathbb{N};(1)$$

$$a_0=a_1=0\Rightarrow a_2=1,a_3=3\overset{(1)}{\Rightarrow}A=\frac{1}{6},B=0,C=-\frac{1}{6},D=0\Rightarrow$$

$$a_n=\frac{1}{6}n(n^2-1)\Rightarrow\frac{n(n-1)}{6a_n}=\frac{1}{n+1},n\geq 0$$

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{k^2(k^2-1)}{18a_ka_{k+1}}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(1-\frac{2k(k-1)\cdot k(k+1)}{6a_k\cdot 6a_{k+1}}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=0}^{n}log\left(1-\frac{2}{(k+1)\cdot(k+2)}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}log\left(\frac{k(k+3)}{(k+1)(k+2)}\right)=$$

$$=\lim_{n\to\infty}log\left(\prod_{k=1}^{n}\frac{k}{k+1}\cdot\frac{k+3}{k+2}\right)=\lim_{n\to\infty}log\left(\frac{n+3}{3(n+1)}\right)=log\left(\frac{1}{3}\right)=-log3$$