R M M

ROMANIAN MATHEMATICAL MAGAZINE

www.ssmrmh.ro

LEGENDRE FORMULA IN TERMS OF THE PRIME COUNTING FUNCTION-REVISITED

By Mohammed Bouras

University Sidi Mohamed Ben Abdellah, Faculty of Science, Fès-Morocco

Abstract:

Let $\pi(x)$ be the prime counting function that gives the number of primes less than or equal to x, Legendre conjectured an approximation which was very similar to $\pi(x)$. We propose an exact representation for the Legendre formula which is valid for infinitely many naturals numbers x. Our proof relies on using the expansion Taylor series at

$x \to \infty$ for some new representations of the Legendre formula

We present the first representation :

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2}}$$

And the second representation :

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2}}$$

$\boldsymbol{where\ the\ Legendre\ constant\ \ a = 1.08633,,,}$

Keywords : Prime number, prime counting function, Legendre formula, Taylor series.

1. Introduction

This paper concerns some representations for the Legendre formula and the Gauss function for the prime counting function $\pi(x)$.

The prime density function, the gauss conjecture, states that :

$$\pi(x) \sim \frac{x}{\ln(x)} \quad as\ \ x \to \infty \qquad (1)$$

The Legendre's conjecture regarding the $\pi(x)$ states that :

$$\pi(x) = \frac{x}{\ln(x) - a} \qquad (2)$$

$where\ the\ Legendre\ constant\ \ a = 1.08633,„$

***Theorem* 1 : we will present now the first representation and the proof for the Legendre formula**

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2} \quad as\ \ x \to \infty}$$

Proof theorem 1 . Taylor series $of\ \pi(x)\ as\ \ x \to \infty$

I'm required to make a Taylor series expansion of a function $\pi(x)$ at $x \to \infty$ .In order to do this I introduce new variable $x = \frac{1}{y}$, so that $x \to +\infty$ is the same as $y \to 0^{+}$. Thus I can expand $\pi\left(\frac{1}{y}\right)$ at $y = 0$:

$$\pi\left(\frac{1}{y}\right) = \frac{1}{y(-\ln(y) - a)} - \frac{1}{2} - \frac{y(\ln(y) + a)}{12} + O(\,y^2)$$

Taylor expansion at infinity

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{1}{2} + \frac{\ln(x) - a}{12x} + O\left(\frac{1}{x^2}\right)$$

We have

$$\lim_{x\to\infty} \frac{\ln(x) - a}{12x} = 0$$

So that

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{1}{2} \quad as\ \ x \to \infty$$

Hence

$$\frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2} = \frac{x}{\ln(x) - a} \qquad (3)$$

The relation (3) give the new representation of the Legendre formula for $a = 1.08633,„$

***Theorem* 2 : The second representation for the Legendre formula**

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2} \quad as \ x \to \infty}$$

Proof theorem 2 .

We use the same method of proof 1 ,Taylor series $of\ \pi(x)\ as\ \ x \to \infty$ and we find:

$$\pi(x) = \frac{x}{\ln(x) - a} - \frac{\ln(x)^2}{2(a - \ln(x))^2} + O\left(\frac{1}{x^2}\right)$$

We are looking for the Taylor series of $\pi(x)$, We have :

$$\frac{\ln(x)^2}{2(a - \ln(x))^2} \sim \frac{1}{2} \quad as \ \ x \to \infty$$

Hence

$$\frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} = \frac{x}{\ln(x) - a} - \frac{1}{2} \quad as \ \ x \to \infty \qquad (4)$$

2. Main results :

Looking at the expression of $(1), (3)$ and (4) above, for $a = 0$ we obtain a new representation for the Gauss conjecture :

$$\boldsymbol{\pi(x) \sim \frac{1}{\sqrt[x]{x} - 1} + \frac{1}{2} \quad as \ \ x \to \infty}$$

Looking at the expression of (2) , (3) and (4) above, for $a = 1.08633,,,$ we obtain some new representations for the legendre formula :

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2}}$$

And

$$\boldsymbol{\pi(x) = \frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2}}$$

This completes the proof. The table shows how the functions $\frac{1}{\sqrt[x]{x} - 1 - \frac{a}{x}} + \frac{1}{2}$, $\frac{1}{\sqrt[x]{x.e^{-a}} - 1} + \frac{1}{2}$ and the Legendre formula compart at powers of 10. $(for\ \ a = 1.08633,,,)$

x	$\frac{1}{\sqrt[x]{x}-1-\frac{a}{x}}+\frac{1}{2}$	$\frac{1}{\sqrt[x]{x.e^{-a}}-1}+\frac{1}{2}$	$\frac{x}{\ln(x)-a}$
10	7	8	8
10^2	28	28	28
10^3	171	171	171
10^4	1230	1230	1230
10^5	9590	9590	9590
10^6	78 559	78 559	78 559
10^7	665 257	665 257	665 257
10^8	5 768 892	5 768 892	5 768 892
10^9	50 924 441	50 924 441	50 924 441
10^{10}	455 798 466	455 798 466	455 798 466
10^{11}	4 125 054 147	4 125 054 147	4 125 054 147
10^{12}	37 672 316 307	37 672 316 307	37 672 316 307
10^{13}	346 653 178 885	346 653 178 885	346 653 178 885
10^{14}	3 210 287 167 276	3 210 287 167 276	3 210 287 167 276
10^{15}	29 893 179 954 460	29 893 179 954 460	29 893 179 954 460
10^{16}	279 680 917 170 575	279 680 917 170 575	279 680 917 170 575
10^{17}	2 627 594 920 124 090	2 627 594 920 124 090	2 627 594 920 124 090
10^{18}	24 776 883 130 563 108	24 776 883 130 563 108	24 776 883 130 563 108
10^{19}	234 396 314 864 306 897	234 396 314 864 306 897	234 396 314 864 306 897
10^{20}	2 223 933 570 740 069 490	2 223 933 570 740 069 490	2 223 933 570 740 069 490

Table 1

We can see above at table 1 that the new representations gives exactly the values of Legendre's formula

REFERENCES :

[1]: *R. Farhadian and R. Jakimczuk, One more disproof for the Legendre's conjecture reading the prime counting function, Notes on Number theory and Discrete Mathematics, DOI*: 10.7546/*nntdm*.2018.24.3.84 – 91

[2]: İ. *Okumuş and E. Çelik, Improved of Approximating Function Li*(x), *International Journal of Engineering and Applied Sciences* (*IJEAS*), *ISSN*: 2394 – 3661, *Volume* – 5, *Issue* – 4, *April* 2018

[3]: *T. Kotnik, The prime counting function and its analytic approximations, Springer science, Slovenia* (2007).

[4]: $A. E. Patkowski, A\ note\ on\ the\ Gram\ Series\ , International\ Mathematical\ Forum$ (2008).

[5]: $J. R. Sousa, An\ Exact\ Formula\ for\ the\ prime\ counting\ function, Reaserchgate, S. Paulo - Brazil$ (2019).

[6]: $M. Hassani, On\ the\ means\ of\ the\ values\ of\ prime\ counting\ function, Iranian\ journal\ of\ mathematical\ sciences\ and\ informatics, Iran$ (2018).