AN USEFUL TRIGONOMETRIC SUBSTITUTION TO PROVE ALGEBRAIC INEQUALITIES

By Marius Drăgan, Neculai Stanciu – Romania

Abstract: This paper presents a method to solve certain algebraic inequalities using geometric inequalities.

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MSC: 08A20, 51M16, 26D05

Main results

I. The refinement of the inequality $\prod(x+y)^2 \geq 4\prod(x^2+yz)$, (Vasile Cârtoaje).

If $x, y, z \geq 0$, then holds the following inequality;

$$(x+y)^2(y+z)^2(z+x)^2 \geq 4(x^2+yz)(y^2+zx)(z^2+xy)+32x^2y^2z^2$$

Proof. If we denote $u=\frac{x}{\alpha}, v=\frac{y}{\alpha}, w=\frac{z}{\alpha}$, where $\alpha=\sqrt{\sum xy}, \sum uv=1$, then the inequality from the statement is successively equivalent to

$$\prod(x+y)^2 \geq 4\left(2x^2y^2z^2+\sum x^3y^3+xyz\sum x^3\right)+32x^2y^2z^2 \Leftrightarrow$$

$$\Leftrightarrow \prod(x+y)^2 \geq 40x^2y^2z^2+4\sum x^3y^3+4xyz\sum x^3$$

$$\Leftrightarrow \prod(u+v)^2 \geq 40u^2v^2w^2+4\sum u^3v^3+4uvw\sum u^3$$

Since we have $\sum uv=1$ there exists the triangle ABC with $u=\tan\frac{A}{2}, v=\tan\frac{B}{2}, w=\tan\frac{C}{2}$ and also are well-known the identities $\prod\tan\frac{A}{2}=\frac{r}{s}, \prod\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=\frac{4R}{s}$

$$\prod\tan^3\frac{A}{2}=\frac{(4R+r)^3}{s^3}-\frac{12R}{s}, \prod\tan^3\frac{A}{2}\tan^3\frac{B}{2}=1-\frac{12Rr}{s^2}$$

Hence, the last inequality becomes successively

$$\frac{4R^2}{s^2} \geq \frac{10r^2}{s^2} + 1 - \frac{12Rr}{s^2} + \frac{r}{s}\left(\frac{(4R+r)^3}{s^3} - \frac{12R}{s}\right) \Leftrightarrow \frac{4R^2 + 24Rr - 10r^2}{s^2}$$

$$\geq 1 + \frac{r(4R+r)^3}{s^4}$$

We denote $u = \frac{1}{s^2}, x = \frac{R}{r}$ and we consider the function

$$f: \left[\frac{1}{4R^2 + 4Rr + 3r^2}, \frac{1}{16Rr - 5r^2}\right] \to R,$$

$$f(u) = r(4R+r)^3u^2 - (4R^2 + 24Rr - 10r^2)u + 1$$

The last inequality is equivalent to $f(u) \leq 0, \forall u \in [u_1, u_2] = \left[\frac{1}{4R^2+4Rr+3r^2}, \frac{1}{16Rr-5r^2}\right]$

So, it remains to prove that $f\left(\frac{1}{16Rr-5r^2}\right) \leq 0$ and $f\left(\frac{1}{4R^2+4Rr+3r^2}\right) \leq 0$.

$f\left(\frac{1}{16Rr-5r^2}\right) \leq 0 \Leftrightarrow 5x^2 - 11x + 2 \geq 0 \Leftrightarrow (x-2)(5x-1) \geq 0$, true since $x \geq 2$.

$$f\left(\frac{1}{4R^2 + 4Rr + 3r^2}\right) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow (4x+1)^3 - (4x^2 + 24x - 10)(4x^2 + 4x + 3) + (4x^2 + 4x + 3)^2 \leq 0$$

$\Leftrightarrow 4x^3 - 5x^2 - x - 10 \geq 0 \Leftrightarrow (x-2)^2(4x^2 + 3x + 2) \geq 0$, which is true.

The proof is complete.

II. If $x, y, z \geq 0$, then the following inequality holds:

$$\sum x^4(y+z) + 2xyz\sum xy \geq \sum x^3(y^2 + z^2) + 2xyz\sum x^2$$

Proof. Denoting $u = \frac{x}{\alpha}, v = \frac{y}{\alpha}, w = \frac{z}{\alpha}$, where $\alpha = \sqrt{\sum xy}, \sum uv = 1$, then the inequality is successively equivalent to $\sum x^4\,(y+z) + 2xyz\sum xy \geq \sum x^3\,(y^2+z^2) + 2xyz\sum x^2$

$$\Leftrightarrow \sum x^4\left(\sum x - x\right) + 2xyz\sum xy \geq \sum x^3\left(\sum x^2 - x^2\right) + 2xyz\sum x^2 \Leftrightarrow$$

$$\Leftrightarrow \sum x^4\sum x + 2xyz\sum xy \geq \sum x^3\sum x^2 + 2xyz\sum x^2 \Leftrightarrow$$

$$\Leftrightarrow \sum u^4\sum u + 2uvw\sum uv \geq \sum u^3\sum u^2 + 2uvw\sum u^2$$

Since we have $\sum uv = 1$ there exists the triangle ABC with $u = \tan\frac{A}{2}, v = \tan\frac{B}{2}, w = \tan\frac{C}{2}$

Using the identities

$$\sum u=\frac{4R+r}{s}, uvw=\frac{r}{s}, \sum x^4=2-\frac{16R(4R+r)}{s^2}+\frac{(4R+r)^4}{s^4}$$

$$\sum x^3=\frac{(4R+r)^3}{s^3}-\frac{12R}{s}, \sum x^2=\frac{(4R+r)^2}{s^2}-2,$$

the last inequality becomes

$$\left(2-\frac{16R(4R+r)}{s^2}+\frac{(4R+r)^4}{s^4}\right)\left(\frac{4R+r}{s}\right)+2\cdot\frac{r}{s}\geq$$

$$\geq\left(\frac{(4R+r)^3}{s^3}-\frac{12R}{s}\right)\left(\frac{(4R+r)^2}{s^2}-2+\frac{2r}{s}\left(\frac{(4R+r)^2}{s^2}-2\right)\right),$$

which after some algebra is equivalent to $p^2\leq\frac{R(4R+r)^2}{4R-2r}$, i.e. Bilcev's inequality.

Remark 1. The inequality can be written as

$$(x-y)^2xy(x+y-z)+(y-z)^2yz(y+z-x)+(z-x)^2zx(z+x-y)\geq 0$$

III. If $x,y,z\geq 0$, then the following inequality holds:

$$\frac{8xyz}{\prod(x+y)}+\frac{(\sum xy)^2}{xyz\sum x}\geq 4$$

(The American Mathematical Monthly).

Proof. We denote $u=\frac{x}{\alpha}, v=\frac{y}{\alpha}, w=\frac{z}{\alpha}$, where $\alpha=\sqrt{\sum xy}, \sum uv=1$. Since we have $\sum uv=1$ we shall consider the triangle ABC with $u=\tan\frac{A}{2}, v=\tan\frac{B}{2}, w=\tan\frac{C}{2}$.

Using the identities $\sum uv=1, \sum u=\frac{4R+r}{s}, uvw=\frac{r}{s}, \prod(u+v)=\frac{4R}{s}$, then the inequality to prove is successively equivalent to

$$\frac{8uvw}{\prod(u+v)}+\frac{(\sum uv)^2}{uvw\sum u}\geq 4\Leftrightarrow\frac{8\cdot\frac{r}{s}}{\frac{4R}{s}}+\frac{1}{\frac{r}{s}\cdot\frac{4R+r}{s}}\geq 4\Leftrightarrow\frac{2r}{R}+\frac{s^2}{r(4R+r)}\geq 4\Leftrightarrow$$

$$\Leftrightarrow s^2\geq\frac{(4R-2r)\cdot r\cdot(4R+r)}{R}$$

In [1] it is proved that $s^2\geq\frac{r(16R^2-20Rr+3r^2)}{R-r}$.

So, if we denote $\frac{R}{r}=x$ it suffices to show that

$$\frac{16x^2-20x+3}{x-1} \geq \frac{(4x-2)(4x+1)}{x} \Leftrightarrow$$

$$\Leftrightarrow 16x^3-20x^2+3x \geq (x-1)(16x^2+4x-8x-2) \Leftrightarrow$$

$\Leftrightarrow 16x^3-20x^2+3x \geq 16x^3-4x^2-2x-16x^2+4x+2 \Leftrightarrow x \geq 2 \Leftrightarrow R \geq 2r$, i.e.

Euler's inequality. The proof is complete.

Remark 2. For other inequality solved by this method see for e.g. [2].

References:

[1] M. Drăgan, I.V. Maftei, S. Rădulescu, ***Some consequences of Blundon inequality***, Gazeta Matematică, No. 1, 2010, 3-9

[2] M. Drăgan, I.V. Maftei, S. Rădulescu, ***Inegalități Matematice (Extinderi și generalizări). Tehnici și metode de demonstrație și rafinare***, Editura Didactică și Pedagogică, 2012.