The law of reciprocity of light between two spheres

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Abstract

We study the problem of finding the percentage of illumination of one sphere by an another sphere in 3-dimensional space and vice versa. The so-called law of reciprocity of light between two spheres states that the sum of these two percents is always 100%. Two proofs are provided.

1 Introduction

Let us consider two disjoint spheres $S_1(O_1, R_1)$ and $S_2(O_2, R_2)$ in 3-dimensional space such that $R_1+R_2 < d$; where $d = |O_1O_2|$ is the central distance and R_1,R_2 are radii of spheres. We assume that centers of both spheres are fixed.

In the beginning, none of these spheres not emit light.

Let the first sphere start to emit light through its surface onto the second sphere. We assume that light spreads radially through rays, in all directions.

Let the number $P_{1,2}$ denote the area of the illuminated surface of the second sphere, and let the number P_2 denote the total area of the second sphere.

After some time, the first sphere stops to emit light and the second sphere starts to emit light onto the first sphere. Similarly, the number $P_{2,1}$ denotes the area of the illuminated surface of the first sphere, and the number P_1 denotes the total area of the first sphere.

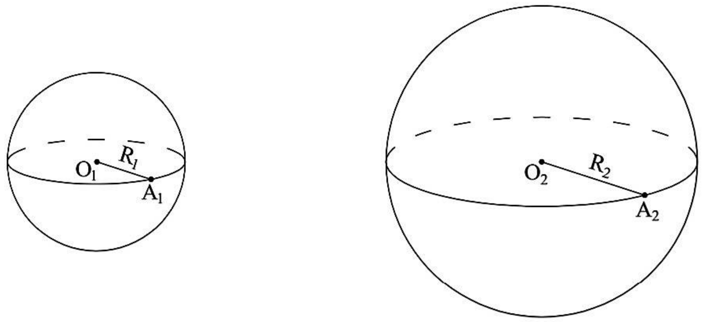

Figure 1:

Our main result is the following theorem:

Theorem 1. (*The law of reciprocity of light between two spheres*)

$$\frac{P_{1,2}}{P_2} + \frac{P_{2,1}}{P_1} = 1. \tag{1}$$

Let the number $p_{1,2}\%$ denote the percentage of illumination of the second sphere by the first sphere, and let the number $p_{2,1}\%$ denote the percentage of illumination of the first sphere by the second sphere. Since $p_{1,2} = \frac{P_{1,2}*100}{P_2}$ and $p_{2,1} = \frac{P_{2,1}*100}{P_1}$, Theorem 1 is equivalent to the following equation

$$p_{1,2}\% + p_{2,1}\% = 100\%. \tag{2}$$

The problem of finding the percentage of a sphere that can be viewed from an external point is a well-known [1, 2]. The problem of finding the percentage of a sphere that can be viewed from an another sphere seems to be a less-known.

We give two proofs of Theorem 1. The similarity of triangles is used in both proofs. The first proof relies on formula for the area of a spherical cap. The second proof relies on the formula for the percentage of a sphere that can be viewed from an external point [1, 2].

There is a possibility that our Theorem 1 is a new result.

2 The First Proof of Theorem 1

Let the first sphere emit light onto the second sphere, and let π be an arbitrary plane such that contains centers of the both spheres.

In that case, let $K_1(O_1, C) = \pi \cap S(O_1, R_1)$ and $K_2(O_2, A) = \pi \cap S(O_2, R_2)$.

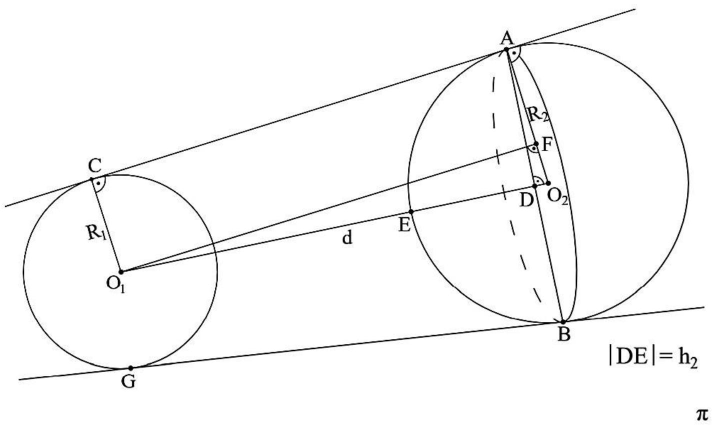

Figure 2:

The circle $K_1(O_1, C)$ will illuminate the arc $\widehat{AB}$ (such that $E \in \widehat{AB}$ or closer to the $K_1(O_1, C)$) of the circle $K_2(O_2, A)$, where CA and GB are common external tangents of those two circles.

Note that if $R_1 < R_2$, then the illuminated arc $\widehat{AB}$ is a smaller one as it is shown on Figure 2. If $R_1 > R_2$, then the illuminated arc is a greater one. Finally, if $R_1 = R_2$, then the illuminated arc $\widehat{AB}$ is half of the circumference of the second circle $K_2(O_2, A)$.

Let the plane π start to rotate around the line that passes through centers O_1 and O_2 of the both spheres. Then the smaller arc $\widehat{AB}$ of the circle $K(O_2, A)$ describes a spherical cap of the second sphere which height is $h_2 = DE$.

Also note that if $R_1 < R_2$, then $0 < h_2 < R_2$ as it is shown on Figure 2. If $R_1 > R_2$, then $R_2 < h_2 < 2R_2$. Finally, if $R_1 = R_2$, then $h_2 = R_2$.

We assume that numbers R_1, R_2, and d are known. Let us determine the height h_2 as a function of R_1, R_2, and d.

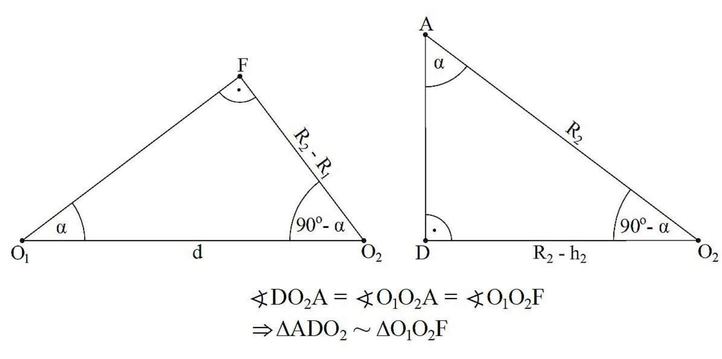

Figure 3:

From the similarity of triangles $\triangle DO_2A$ and $\triangle ADO_2$, we have gradually

$$\begin{aligned}
\frac{R_2-h_2}{R_2-R_1} &= \frac{R_2}{d} \\
R_2(R_2-R_1) &= d(R_2-h_2) \\
R_2(R_2-R_1) &= dR_2 - dh_2 \\
dh_2 &= dR_2 + R_2(R_1-R_2) \\
h_2 &= \frac{R_2(d+R_1-R_2)}{d}
\end{aligned}$$

From the last equation above, we obtain that

$$\frac{h_2}{R_2} = \frac{d+R_1-R_2}{d}. \tag{3}$$

By dividing two well-known formulas [3],

$$\begin{aligned}
P_{1,2} &= 2R_2\pi h_2 \\
P_2 &= 4R_2^2\pi
\end{aligned}$$

we obtain that

$$\frac{P_{1,2}}{P_2} = \frac{h_2}{2R_2}. \tag{4}$$

By Eqns. (3) and (4), we conclude that

$$\frac{P_{1,2}}{P_2} = \frac{d+R_1-R_2}{2d}. \tag{5}$$

By the analogy, it is not hard to prove that

$$\frac{P_{2,1}}{P_1} = \frac{d+R_2-R_1}{2d}. \tag{6}$$

Finally, by adding Eqns. (5) and (6), we obtain the Eq. (1).

This completes the proof of Theorem 1.

Remark 2. The law of reciprocity holds in 2-dimensional space too.

Let $K_1(0_1, C)$ and $K_2(0_2, A)$ be two circles in an arbitrary plane π, as it shown on Figure 2.

As it said before, the first circle $K_1(0_1, C)$ illuminate the arc $\widehat{AB}$ (a smaller one or closer to the first circle) of the second circle.

Similarly, the second circle $K_2(0_2, A)$ illuminate the arc $\widehat{GC}$ (a greater one or closer to the second circle) of the first circle.

It is readily verified that

$$\frac{|\widehat{AB}|}{2R_2\pi} + \frac{|\widehat{GC}|}{2R_1\pi} = 1. \tag{7}$$

Remark 3. If we set $R_1 = 0$, then the first sphere reduces to the center O_1 and the Eq. (5) becomes

$$\frac{P_{1,2}}{P_2} = \frac{d - R_2}{2d}. \tag{8}$$

It is readily verified that the Eq. (8) is the formula for the percentage of the second sphere $S_2(O_2, R_2)$ that can be viewed from the point O_1 [1, 2].

3 The Second Proof of Theorem 1

If $R_1 = R_2$, then the first sphere illuminates exactly 50% of the another sphere and vice versa. In this case, Theorem 1 easily follows.

Let us assume that $R_1 \neq R_2$. Without loss of generality, let $R_1 < R_2$.

In that case the external center of similarity for spheres $S_1(O_1, R_1)$ and $S_2(O_2, R_2)$ exists and we denote it by point O.

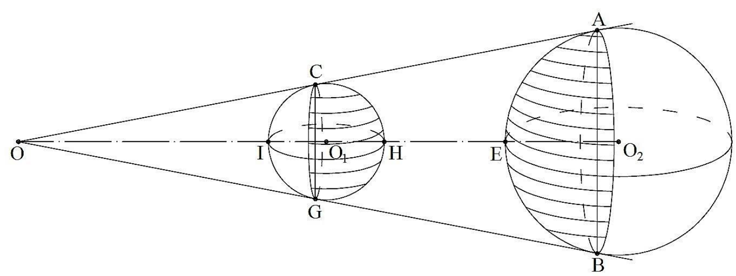

Figure 4:

The illuminated surface of the first sphere by the second sphere is equal to the non-visible fraction of the first sphere from the point O. It is represented as a shaded spherical cap of the first sphere on Figure 4.

The illuminated surface of the second sphere by the first sphere is equal to the visible fraction of the second sphere from the point O (we assume that the first sphere is removed, and there are no obstacles between the point O and the second sphere). It is represented as a shaded spherical cap of the second sphere on Figure 4.

Let $P_{O_1,1}$ and $P_{O_1,2}$ denote the visible fraction of the first and the second sphere from the point O, respectively.

Then we have

$$P_{1,2} = P_{O_1,2} \tag{9}$$
$$P_{2,1} = 1 - P_{O_1,1}. \tag{10}$$

Let d_1 denote $|OO_1|$ and let d_2 denote $|OO_2|$.

By the Eq. (8), we know that

$$P_{O_1,2} = \frac{d_2 - R_2}{2d_2} \tag{11}$$

$$P_{O_1,1} = \frac{d_1 - R_1}{2d_1}. \tag{12}$$

See also [1, 2].
Let π be an arbitrary plane such that contains centers of the both spheres. Again let $K_1(O_1, C) = \pi \cap S(O_1, R_1)$, $K_2(O_2, A) = \pi \cap S(O_2, R_2)$; where $O - C - A$ and OA is the common external tangent for those circles.

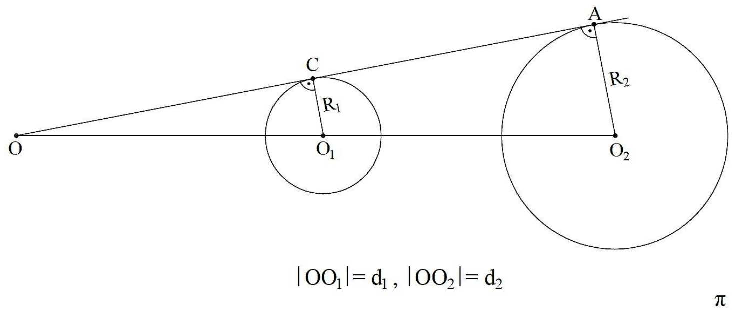

Figure 5:

By similarity of triangles $\triangle OO_1C$ and $\triangle OO_2A$, we have

$$\frac{R_1}{d_1} = \frac{R_2}{d_2} = k. \tag{13}$$

By Eqns. (11), (12), and (13) it follows that

$$P_{O_1,2} = \frac{1-k}{2} = P_{O_1,1}.$$

By the last equation above, we obtain that

$$P_{O_1,2} = P_{O_1,1}. \tag{14}$$

Finally, by Eqns. (9), (10), and (14), it follows that

$$\begin{aligned} P_{1,2} + P_{2,1} &= P_{O_1,2} + (1 - P_{O_1,1}) \quad \text{(by Eqns. (9) and (10))} \\ &= P_{O_1,1} + (1 - P_{O_1,1}) \quad \text{(by the Eq. (14))} \\ &= 1. \end{aligned}$$

The last equation above completes the second proof of Theorem 1.

References

[1] https://math.stackexchange.com/questions/1329130/what-fraction-of-a-sphere-can-an-external-observer-see/1329155

[2] https://www.quora.com/How-much-of-a-spherical-planet-can-one-see-at-once

[3] https://en.wikipedia.org/wiki/Spherical$_{c}ap$