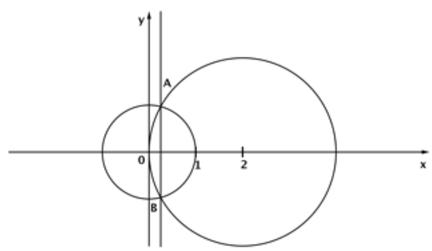
SOLVED PROBLEMS FROM CRUX MATHEMATICORUM

DANIEL SITARU - ROMANIA

CC203. Two circles, one of radius 1, the other of radius 2, intersect so that the larger circle passes through the centre of the smaller circle. Find the distance between the two points at which the circles intersect.

Proof.



$$O_1(0,0); O_2(1,0)$$

$$C_1 : x^2 + y^2 = 1; C_2 : (x-2)^2 + y^2 = 4$$

The radical axis:

$$AB : x^2 - 4x + 4 + y^2 - 4 - x^2 - y^2 + 1 = 0$$

$$AB : x = \frac{1}{4}$$

$$\left(\frac{1}{4}\right)^2 + y^2 = 1 \Rightarrow y^2 = \frac{15}{16} \Rightarrow y = \pm\frac{\sqrt{15}}{4}$$

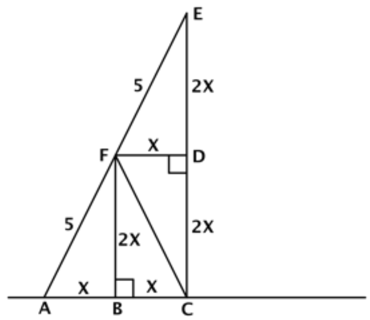
$$A\left(\frac{1}{4}, \frac{\sqrt{5}}{4}\right); B\left(\frac{1}{4}; -\frac{\sqrt{15}}{4}\right)$$

$$AB = 2 \cdot \frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{2}$$

□

CC204. A 10 metre ladder rest against a vertical wall. The midpoint of the ladder is twice as far from the ground as it is form the wall. At what height on the wall does the ladder reach?

Proof.



$$(2x)^2 + x^2 = 5^2 \Rightarrow 5x^2 = 5^2 \Rightarrow x^2 = 5 \Rightarrow x = \sqrt{5}$$
$$CE = 4x = 4\sqrt{5}$$

□

CC218. Solve the following system of equations:

$$\begin{cases} 3^{\ln x} = 4^{\ln y} \\ (4x)^{\ln 4} = (3y)^{\ln 3} \end{cases}$$

Proof.

$$\begin{cases} 3^{\ln x} = 4^{\ln y} \\ (4x)^{\ln 4} = (3y)^{\ln 3} \end{cases} \Rightarrow \begin{cases} \ln 3^{\ln 3} = \ln 4^{\ln y} \\ \ln(4x)^{\ln 4} = \ln(3y)^{\ln 3} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \ln x \ln 3 = \ln y \ln 4 \\ \ln 4(\ln 4 + \ln x) = \ln 3(\ln 3 + \ln y) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \ln y = \frac{\ln x \ln 3}{\ln 4} \\ \ln 4(\ln 4 + \ln x) = \ln^2 3 + \ln 3 \cdot \frac{\ln x \ln 3}{\ln 4} \end{cases}$$

$$\ln x \ln 4 + \ln^2 4 = \frac{\ln^2 3 \ln 4 + \ln x \ln^2 3}{\ln 4}$$
$$\ln x(\ln^2 4 - \ln^2 3) + \ln^3 4 - \ln^2 3 \ln 4 = 0$$
$$\ln x(\ln^2 4 - \ln^2 3) + \ln 4(\ln^2 4 - \ln^2 3) = 0$$
$$(\ln^2 4 - \ln^2 3)(\ln x + \ln 4) = 0$$
$$\ln x = -\ln 4 \Rightarrow x = \frac{1}{4}$$
$$\ln y = \frac{-\ln 4 \ln 3}{\ln 4} = -\ln 3 \Rightarrow y = \frac{1}{3}$$

Generalization 1:
Solve the following system of equations:

$$\begin{cases} a^{\ln x} = b^{\ln y} \\ (bx)^{\ln b} = (ay)^{\ln a} \end{cases} ; a, b \in (0,1) \cup (1,\infty); a \neq b$$

Proof.

$$x = \frac{1}{b}; y = \frac{1}{a}$$

□

Generalization 2:
Solve the following system of equations:

$$\begin{cases} a^{\log_c x} = b^{\log_c y} \\ (bx)^{\log_c b} = (ay)^{\log_c b} \end{cases} ; a, b, c \in (0,1) \cup (1, \infty); a \neq b \neq c \neq a$$

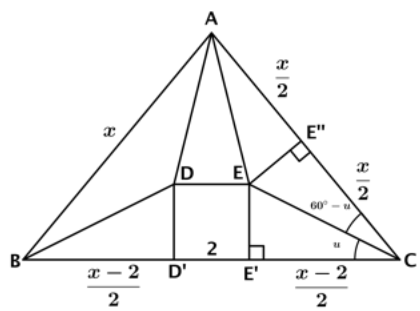
Proof.

$$x = \frac{1}{b}; y = \frac{1}{a}$$

□

□

CC222. Quelle est la valeur de x dans la figure plane suivante?



Proof. In $\Delta EE'C : \cos u = \frac{E'C}{EC} \Rightarrow \cos u = \frac{\frac{x-2}{2}}{7}$

$$7 \cos u = \frac{x-2}{2} \Rightarrow x - 2 = 14 \cos u \Rightarrow x = 2 + 14 \cos u \tag{0.1}$$

In $\Delta EE''C : \cos(60^\circ - u) = \frac{E''C}{EC} \Rightarrow \cos(60^\circ - u) = \frac{\frac{x}{2}}{7}$

$$14 \cos(60^\circ - u) = x \tag{0.2}$$

By 0.1; 0.2:

$$2 + 14 \cos u = 14 \cos(60^\circ - u)$$

$$14\left(\frac{1}{2} \cos u + \frac{\sqrt{3}}{2} \sin u\right) = 14 \cos u + 2$$

$$7 \cos u + 7\sqrt{3} \sin u = 14 \cos u + 2$$

$$7\sqrt{3} \sin u - 7 \cos u = 2 \Rightarrow 7 \sin\left(u - \frac{\pi}{6}\right) = 1$$

$$\sin\left(u - \frac{\pi}{6}\right) = \frac{1}{7} \Rightarrow u - \frac{\pi}{6} = \sin^{-1}\left(\frac{1}{7}\right)$$

$$u = \frac{\pi}{6} + \sin^{-1}\left(\frac{1}{7}\right)$$

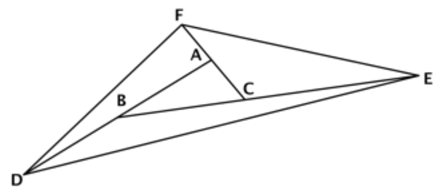
$$x = 14 \cos\left(\frac{\pi}{6} + \sin^{-1}\left(\frac{1}{7}\right)\right) + 2 = 14\left(\frac{\sqrt{3}}{2} \cdot \sqrt{1 - \frac{1}{49}} + \frac{1}{2} \cdot \frac{1}{7}\right) + 2 =$$

$$= \sqrt{3} \cdot 4\sqrt{3} + 3 = 15$$

□

CC225

The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{2}AB$; $CE = \frac{1}{2}BC$ and $AF = \frac{1}{2}CA$. What is the ratio of the area of triangle DEF to that of triangle ABC?



Proof.

$$\sphericalangle DBE = \sphericalangle A + \sphericalangle C = \pi - \sphericalangle B$$
$$[DBE] = \frac{1}{2}BD \cdot BE \cdot \sin(\widehat{DBE}) =$$
$$= \frac{1}{2}\left(c + \frac{c}{2}\right)\left(a + \frac{a}{2}\right)\sin(\pi - \sphericalangle B) =$$
$$= \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}ac\sin B = \frac{9}{4}[ABC]$$

Analogous

$$[CEF] = \frac{9}{4}[ABC]; [AFD] = \frac{9}{4}[ABC]$$
$$\frac{[DEF]}{[ABC]} = \frac{[ABC] + \frac{9}{3}[ABC] + \frac{9}{4}[ABC] + \frac{9}{4}[ABC]}{[ABC]}$$
$$= 1 + \frac{9}{4} + \frac{9}{4} + \frac{9}{4} = \frac{31}{4}$$

Generalization:

The three sides of triangle ABC are extended as shown so that $BD = \frac{1}{m}AB$; $CE = \frac{1}{m}BC$ and $AF = \frac{1}{m}CA$; $m \in \mathbb{R}; m > 1$.

What is the ratio of the area of triangle DEF to that of triangle ABC?

Proof.

$$[DBE] = \frac{1}{2}\left(c + \frac{c}{m}\right)\left(a + \frac{a}{m}\right)\sin(\pi - \sphericalangle B) =$$
$$= \frac{(m+1)c}{m} \cdot \frac{(m+1)a}{m} \cdot \frac{1}{2}\sin B =$$
$$= \left(\frac{m+1}{m}\right)^2 \cdot \frac{1}{2}ac\sin B = \left(\frac{m+1}{m}\right)^2 \cdot S$$

Analogous:

$$[CEF] = \left(\frac{m+1}{m}\right)^2[ABC]; [AFD] = \left(\frac{m+1}{m}\right)^2[ABC]$$
$$\frac{[DEF]}{[ABC]} = \frac{[ABC] + \left(\frac{m+1}{m}\right)^2[ABC] + \left(\frac{m+1}{m}\right)^2[ABC] + \left(\frac{m+1}{m}\right)^2[ABC]}{[ABC]} =$$
$$= 1 + 3 \cdot \frac{(m+1)^2}{m^2} = \frac{m^2 + 3m^2 + 6m + 3}{m^2} = \frac{4m^2 + 6m + 3}{m^2}$$

□

□

OC281. Find all polynomials $P(x)$ with real coefficients such that:

$$P(P(x)) = (x^2 + x + 1) \cdot P(x)$$

where $x \in \mathbb{R}$

Proof. $P = 0$ it is a solution. Suppose that $P \neq 0$.
If $gradP = 0, gradP = 1$ no solutions. Suppose that $gradP = n \geq 2$.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in \mathbb{R}[x]; n \in \mathbb{N}$$

$$gradP = n$$

$$\left.\begin{array}{l} gradP(P(x)) = n^2 \\ grad(x^2+x+1)P(x) = n+2 \end{array}\right\} \Rightarrow n^2 = n+2 \Rightarrow n^2 - n - 2 = 0$$

$$\Rightarrow n = 2 \Rightarrow P(x) = ax^2 + bx + c; a \neq 0; a, b, c \in \mathbb{R}$$

$$a(ax^2+bx+c)^2 + b(ax^2+bx+c) + c = (x^2+x+1)(ax^2+bx+c)$$

$$a^3x^4 + 2a^2bx^3 + (ab^2 + 2a^2c + ab)x^2 + (2abc + b^2)x + ac^2 + bc + c =$$

$$= ax^4 + (b+a)x^3 + (c+a+b)x + c$$

$$(S): \begin{cases} a^3 = a \\ 2a^2 b = b + a \\ ab^2 + 2a^2c + ab = c + a + b \\ 2abc + b^2 = b + c \\ ac^2 + bc = 0 \end{cases}$$

For $a = 1 \Rightarrow 2b = b + 1 \Rightarrow b = 1; 2c + 1 = 1 + c \Rightarrow P(x) = x^2 + x$
For $a = -1 \Rightarrow b = -1 \Rightarrow c = -2$ but these values don't verify the last equation of (S). The only polynome is $P(x) = x^2 + x$. □

OC282. Let x, y, z be three nonzero real numbers satisfying $x + y + z = xyz$. Prove that

$$\sum \Big(\frac{x^2-1}{x}\Big)^2 \geq 4$$

Proof. Let A, B, C be the angles of $\Delta ABC, x = \tan A; y = \tan B; z = \tan C$ because:

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C \Leftrightarrow$$

$$\Leftrightarrow x + y + z = xyz$$

$$\sum \Big(\frac{x^2-1}{x}\Big)^2 \geq 4 \Leftrightarrow \sum \Big(\frac{\tan^2 A - 1}{\tan A}\Big)^2 \geq 4 \Leftrightarrow$$

$$\Leftrightarrow \sum \Big(\frac{\sin^2 A - \cos^2 A}{\sin A \cos A}\Big)^2 \geq 4 \Leftrightarrow \sum \Big(\frac{\cos 2A}{\sin 2A}\Big)^2 \geq 1$$

$$\Leftrightarrow \sum \cot^2 A \geq 1 \text{ (to prove)}$$

$$\sqrt{\frac{\sum \cot^2 A}{3}} \geq \frac{\sum \cot A}{3}$$

$$\sum \cot^2 A \geq \frac{(\sum \cot A)^2}{3} =$$

$$= \frac{1}{3}\Big(\frac{b^2+c^2-a^2}{4S} + \frac{a^2+b^2-c^2}{4S} + \frac{a^2+c^2-b^2}{4S}\Big)^2 =$$

$$= \frac{1}{3}\Big(\frac{a^2+b^2+c^2}{4S}\Big)^2 \geq 1 \text{ (to prove)}$$
$$(a^2+b^2+c^2) \geq 48S^2$$
$$a^2+b^2+c^2 \geq 4\sqrt{3}S$$

which is Ionescu - Weitzenbock's inequality □

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com