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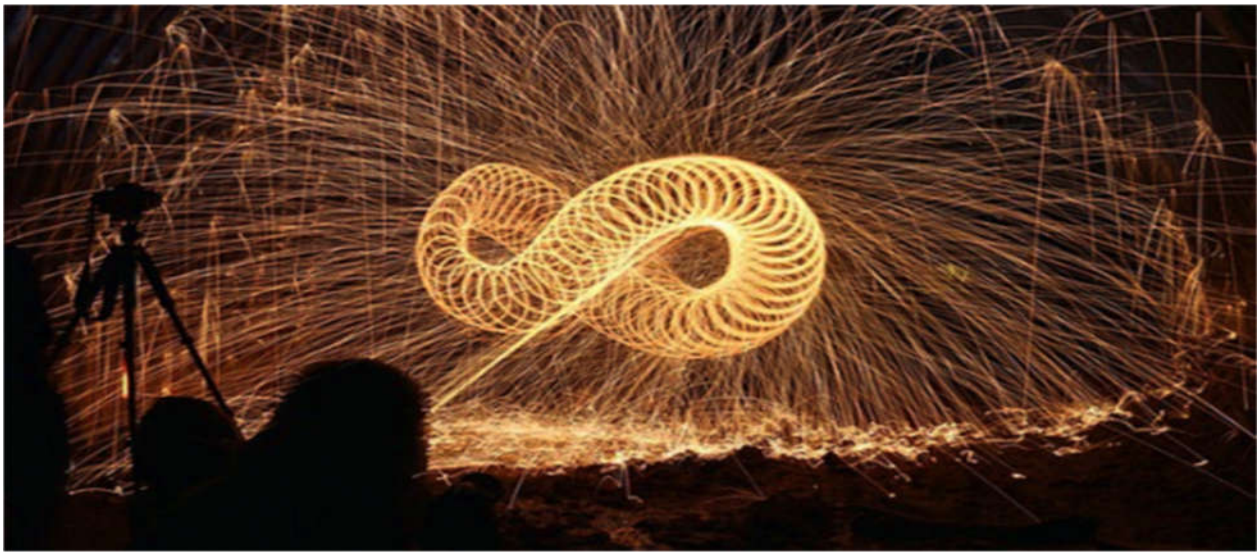

SP.352 Let $(x_n)_{n\ge1}, (y_n)_{n\ge1}$ be sequences of real numbers with

$$x_1 = 0, y_1 = 1,$$

$$x_{n+1} = \frac{ax_n + by_n}{a+b}, y_{n+1} = \frac{cx_n + dy_n}{c+d}, \forall n \ge 1, a, b, c, d > 0, ad \ne bc.$$

Prove that if $(z_n)_{n\ge1}, z_n = y_n - x_n$, then $(z_n)_{n\ge1}$ −geometric progression, and if $q < 1, q$ −ratio of progression, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Proposed by Marin Chirciu-Romania

Solution 1 by proposer, Solution 2 by Remus Florin Stanca-Romania

Solution 1 by proposer

$$z_{n+1} = y_{n+1} - x_{n+1} = \frac{ad-bc}{(a+b)(c+d)}(y_n - x_n) = \frac{ad-bc}{(a+b)(c+d)} z_n$$

We deduce that $(z_n)_{n\ge1}$ −geometric progression with $q = \frac{ad-bc}{(a+b)c+d)}$ ratio, (1).

We get, $z_n = z_1 q^{n-1} = 1 \cdot q^{n-1} = q^{n-1}$, where $q = \frac{ad-bc}{(a+b)c+d)}$.

From $z_n = y_n - x_n$ and $z_n = q^{n-1}$, then $y_n = x_n + q^{n-1}$ which replacing in $x_{n+1} = \frac{ax_n+by_n}{a+b}$, it follows $x_{n+1} = x_n + \frac{b}{a+b} q^{n-1}$, and then for $1, 2, \dots, n-1$ values, we get: $x_n = x_1 + \frac{b}{a+b}(1 + q + q^2 + \cdots + q^{n-1}) = 0 + \frac{b}{a+b} \cdot \frac{1-q^n}{1-q} = \frac{b}{a+b} \cdot \frac{1-q^n}{1-q}$.

How $q \in (0,1)$, it follows that $(x_n)_{n\geq 1}, x_n = \frac{b}{a+b} \cdot \frac{1-q^n}{1-q}$ is convergent, and then

$\lim\limits_{n\to\infty} x_n = \frac{b}{a+b} \cdot \frac{1}{1-q}$ and replacing $q = \frac{ad-bc}{(a+b)c+d)}$, we get:

$$\lim_{n\to\infty} x_n = \frac{b(c+d)}{ac+2bc+bd}.$$

Using $y_n = x_n + q^{n-1}, q \in 0,1), (x_n)_{n\geq 1}$ −convergent, then $(y_n)_{n\geq 1}$ −convergent.

Hence, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$.

Solution 2 by Remus Florin Stanca-Romania

$$z_n = y_n - x_n = \frac{cx_{n-1} + dy_{n-1}}{c+d} - \frac{ax_{n-1} + by_{n-1}}{a+b} =$$

$$= \frac{acx_{n-1} + ady_{n-1} + bcx_{n-1} + bdy_{n-1} - acx_{n-1} - bcy_{n-1} - adx_{n-1} - bdy_{n-1}}{(a+b)(c+d)} =$$

$$= \frac{adz_{n-1} - bcz_{n-1}}{(a+b)(c+d)} = z_n \Rightarrow \frac{z_n}{z_{n-1}} = \frac{ad-bc}{(a+b)(c+d)} = q$$

$$y_{n+1} = \frac{cx_n + dy_n}{c+d} = \frac{c}{c+d}x_n + \frac{d}{c+d}y_n + \frac{c}{c+d}y_n - \frac{c}{c+d}y_n \Rightarrow$$

$$y_{n+1} - y_n = -\frac{c}{c+d}z_n \Rightarrow \sum_{k=1}^{n}(y_{k+1} - y_k) = -\frac{c}{c+d}\sum_{k=1}^{n} z_k \Rightarrow$$

$$y_{n+1} - y_1 = -\frac{c}{c+d}z_1 \cdot \frac{q^n - 1}{q-1}; z_1 = 1 \Rightarrow y_{n+1} = 1 - \frac{c}{c+d} \cdot \frac{q^n - 1}{q-1}; q < 1 \Rightarrow$$

$$\lim_{n\to\infty} y_n = 1 - \frac{c}{c+d} \cdot \frac{1}{1-q}; (1)$$

$$x_{n+1} = \frac{ax_n + by_n}{a+b} = \frac{ax_n}{a+b} + \frac{bx_n}{a+b} + \frac{by_n}{a+b} - \frac{bx_n}{a+b} \Rightarrow x_{n+1} - x_n = \frac{b}{a+b}z_n \Rightarrow$$

$$\sum_{k=1}^{n}(x_{k+1} - x_k) = \frac{b}{a+b} \cdot \frac{q^n - 1}{q-1} \Rightarrow \lim_{n\to\infty} x_n = \frac{b}{a+b} \cdot \frac{1}{1-q}; (2)$$

We need to prove that:

$$1 - \frac{c}{c+d} \cdot \frac{q^n - 1}{q-1} = \frac{b}{a+b} \cdot \frac{1}{1-q} \Leftrightarrow \frac{1}{1-q}\left(\frac{c}{c+d} + \frac{b}{a+b}\right) = 1 \Leftrightarrow$$

$$\frac{ac+2bc+bd}{(a+b)(c+d)} = 1 - q \Leftrightarrow 1 - \frac{ad-bc}{(a+b)(c+d)} = \frac{ac+2bc+bd}{(a+b)(c+d)} \Leftrightarrow$$

$$\frac{ac+ad+bc+bd-ad+bc}{(a+b)(c+d)}=\frac{ac+2bc+bd}{(a+b)(c+d)}$$, **which is true.**

Therefore,

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n.$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.