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**SP.351 If $x, y \in \left(0, \frac{\pi}{2}\right)$; $\sqrt[3]{1+\tan x}+\sqrt[3]{1+\tan y}=2\sqrt[3]{2}$ then:**

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}\le 4-2\sqrt[3]{2}$$

***Proposed by Daniel Sitaru - Romania***

***Solution 1 by proposer, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 3 by Tran Hong-Dong Thap-Vietnam***

## *Solution 1 by proposer*

$$\text{Denote } z=\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}$$

$$z^3=\left(\sqrt[3]{1+\tan x}\right)^3+\left(\sqrt[3]{1-\tan x}\right)^3+$$

$$+3\sqrt[3]{1+\tan x}\cdot\sqrt[3]{1-\tan x}\left(\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}\right)$$

$$z^3=1+\tan x+1-\tan x+3\sqrt[3]{1-\tan^2 x}\cdot z$$

$$\frac{z^3-2}{3z}=\sqrt[3]{1-\tan^2 x}\le 1\Rightarrow\frac{z^3-2}{3z}\le 1$$

$$z^3-2\le 3z\Rightarrow z^3-3z\le 0$$

$$z^3-z-2(z+1)\le 0\Rightarrow z(z-1)(z+1)-2(z+1)\le 0$$

$$(z+1)(z^2-z-2)\le 0$$

$$(z+1)\left(z^2-1-(z+1)\right)\le 0$$

$$(z+1)\left((z-1)(z+1)-(z+1)\right)\le 0$$

$$(z+1)^2(z-2)\le 0\Rightarrow z-2\le 0\Rightarrow z\le 2$$

$$\sqrt[3]{1+\tan x}+\sqrt[3]{1-\tan x}\le 2\ \ (1)$$

**Analogous:**

$$\sqrt[3]{1+\tan y}+\sqrt[3]{1-\tan y}\le 2\ \ \ (2)$$

**By adding (1); (2):**

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}+\sqrt[3]{1+\tan x}+\sqrt[3]{1+\tan y}\le 4$$

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}+2\sqrt[3]{2}\le 4$$

$$\sqrt[3]{1-\tan x}+\sqrt[3]{1-\tan y}\le 4-2\sqrt[3]{2}$$

## *Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco*

**Let** $a=\sqrt[3]{1+tan\ x};b=\sqrt[3]{1+tan\ y}\Rightarrow 1-tan\ x=2-a^3;1-tan\ y=2-b^3$

**So, we need to prove that:** $\sqrt[3]{2-a^3}+\sqrt[3]{2-b^3}\le 4-2\sqrt[3]{2},\forall a,b\in\left(1,2\sqrt[3]{2}-1\right)$

**and** $a+b=2\sqrt[3]{3}$

**We have:** $\sqrt[3]{2-a^3}\le 2-a\Leftrightarrow 2-a^3\le(2-a)^3\Leftrightarrow 6(1-a)^2\ge 0$ **which is true.**

**Hence,**

$\sqrt[3]{2-a^3}\le 2-a$ **and** $\sqrt[3]{2-b^3}\le 2-b$

**Therefore,**

$$\sqrt[3]{2-a^3}+\sqrt[3]{2-b^3}\le 4-(a+b)\le 4-2\sqrt[3]{2}$$

## *Solution 3 by Tran Hong-Dong Thap-Vietnam*

**Let us denote:** $a=\sqrt[3]{1+tan\ x};b=\sqrt[3]{1-tan\ x};c=\sqrt[3]{1+tan\ y};d=\sqrt[3]{1-tan\ y}$

$$\Rightarrow a^3+b^3=2;c^3+d^3=2\Rightarrow a,c<1<b,d$$

**Other,** $\sqrt[3]{1+tan\ x}+\sqrt[3]{1+tan\ y}=2\sqrt[3]{2}\Leftrightarrow b+d=2\sqrt[3]{2}.$

**Inequality becomes as:** $a+c\le 4-2\sqrt[3]{2};(1)$

**If** $a,c<0$ **then:** $0<4-2\sqrt[3]{2}\Leftrightarrow 2\sqrt[3]{2}<4\Leftrightarrow 16<64(true).$

**If $0 \le a, c < 1$ then: $(1) \Leftrightarrow a + c \le 4 - (b + d) \Leftrightarrow a + b + c + d \le 4$, which is true by Holder's inequality:**

$$\frac{a^3}{1} + \frac{b^3}{1} + \frac{c^3}{1} + \frac{d^3}{1} \ge \frac{(a+b+c+d)^3}{4^2} \Rightarrow (a+b+c+d)^3 \le 4^3 \Rightarrow a+b+c+d \le 4$$

**If $a < 0 < c < 1$ then: $a + b + c + d < c + b + d < 1 + 2\sqrt[3]{2} < 4 \Rightarrow (1)$ is true.**

**If $c < 0 < a < 1$ then: $a + b + c + d < a + b + d < 1 + 2\sqrt[3]{2} < 4 \Rightarrow (1)$ is true.**

**Note by editor:**

**Many thanks to Florică Anastase-Romania for typed solutions.**