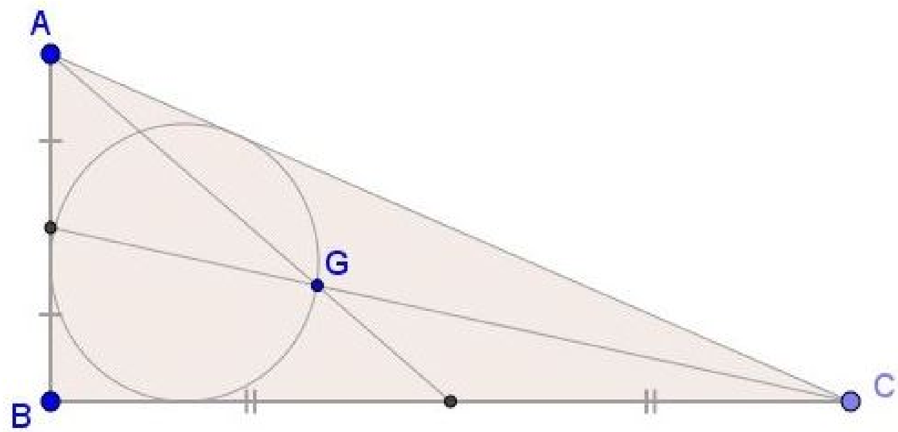

SP.348 Prove that if G-centroid in ΔABC lies on incircle then:

$$s^2 = 16Rr + 4r^2$$

Proposed by Marian Ursărescu-Romania
Solution 1 by proposer, Solution 2 by Daniel Văcaru-Romania

Solution 1 by proposer

The inscribed circle of ΔABC passes through to G if and only if $IG = r \Leftrightarrow IG^2 = r^2$; (1)

$$\overrightarrow{IG} = \frac{\overrightarrow{IA} + \overrightarrow{IB} + \overrightarrow{IC}}{3} \Leftrightarrow IG^2 = \frac{\sum IA^2 + 2\sum \overrightarrow{IA}\cdot\overrightarrow{IB}}{9}; (2)$$

$$\sum IA^2 = s^2 + r^2 - 8Rr; (3)$$

$$\sum \overrightarrow{IA}\cdot\overrightarrow{IB} = \sum \frac{IA^2 + IB^2 - AB^2}{2} = \sum \frac{bc - 4Rr + ac - 4Rr - c^2}{2} =$$

$$= \sum \frac{2c(s-c) - 8Rr}{2} = \sum c(s-c) - 12Rr = 8Rr + 2r^2 - 12Rr = 2r^2 - 4Rr; (4)$$

From (2),(3),(4) it follows that:

$$IG^2 = \frac{s^2 + r^2 - 8Rr + 4r^2 - 8Rr}{9} = \frac{s^2 + 5r^2 - 16Rr}{9}; (5)$$

From (1),(5) it follows that $\frac{s^2+5r^2-16Rr}{9} = r^2 \Leftrightarrow s^2 = 16Rr + 4r^2$

Solution 2 by Daniel Văcaru-Romania

We know that: $9GI^2 = s^2 + 5r^2 - 16Rr$; $(*)$, It follows that ΔABC prove that inscribed circle of ΔABC passes through to G if and only if $GI = r$ and we have equivalency

$$9r^2 = 9GI^2 = s^2 + 5r^2 - 16Rr \Leftrightarrow s^2 = 16Rr + 4r^2.$$

To prove $(*)$, we use Leibniz relationship, i.e. for all point M and ΔABC, we have

$MA^2 + MB^2 + MC^2 = \frac{a^2+b^2+c^2}{3} + 3MG^2$ and $AI = \frac{b+c}{2s}AD$, where AD is bisectors from A.

We have: $AI = \frac{b+c}{2s} \cdot \frac{2bc}{b+c} cos\frac{A}{2} = \frac{bc}{s} cos\frac{A}{2} \Rightarrow AI^2 = \frac{b^2c^2}{s^2} \cdot \frac{s(s-a)}{bc} = \frac{bc(s-a)}{s} \Rightarrow$

$$\sum_{cyc} AI^2 = \frac{s(ab+bc+ca) - 3abc}{s} = ab + bc + ca - 12Rr = s^2 + r^2 - 8Rr$$

We obtain: $9IG^2 = 3(s^2 + r^2 - 8Rr) - (2s^2 - 2r^2 - 8Rr) = s^2 + 5r^2 - 16Rr$

Note by editor:

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