

# R M M

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**JP.360** If  $x, y, z \in \left(0, \frac{\pi}{2}\right)$  then:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \tan x} \geq 2 \sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

*Proposed by Daniel Sitaru – Romania*

*Solution 1 by proposer, Solution 2 by Daniel Văcaru-Romania*

**Solution 1 by proposer**

$$\begin{aligned} \frac{\tan^2 x}{\tan^3 x + \cot x} + \frac{\cot^2 x}{\cot^3 x + \tan x} &\stackrel{\text{BERGSTROM}}{\geq} \frac{(\tan x + \cot x)^2}{\tan^3 x + \cot^3 x + \tan x + \cot x} \\ &= \frac{(\tan x + \cot x)^2}{(\tan x + \cot x)(\tan^2 x - \tan x \cot x + \cot^2 x + 1)} = \\ &= \frac{\tan x + \cot x}{\tan^2 x + \cot^2 x} \stackrel{\text{AM-GM}}{\geq} \frac{2\sqrt{\tan x \cot x}}{\tan^2 x + \cot^2 x} = \\ &= \frac{2}{\tan^2 x + \cot^2 x} \end{aligned}$$

By summing:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \cot x} \geq 2 \sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

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Equality holds for:  $x = y = z = \frac{\pi}{4}$ .

### *Solution 2 by Daniel Văcaru-Romania*

We have  $\frac{\tan^2 x}{\tan^3 x + \cot x} = \frac{\tan^2 x}{\tan^3 x + \frac{1}{\tan x}} = \frac{\tan^3 x}{\tan^4 x + 1}$  and  $\frac{\cot^2 x}{\cot^3 x + \tan x} = \frac{\frac{1}{\tan^2 x}}{\frac{1}{\tan^3 x} + \tan x} = \frac{\tan x}{\tan^4 x + 1}$ .

It follows that:  $\frac{\tan^2 x}{\tan^3 x + \cot x} + \frac{\cot^2 x}{\cot^3 x + \tan x} = \frac{\tan^3 x + \tan x}{\tan^4 x + 1}$ . On the other hand, with calculus, we

find  $\frac{1}{\tan^2 x + \cot^2 x} = \frac{\tan^2 x}{\tan^4 x + 1}$ . We must to prove that:

$$\frac{\tan^3 x + \tan x}{\tan^4 x + 1} \geq \frac{2 \tan^2 x}{\tan^4 x + 1}; (*)$$

But is clear that  $\tan^3 x + \tan x = \tan x (\tan^2 x + 1) \stackrel{AM-GM}{\geq} 2 \tan x$ , which proves (\*).

Writing (\*) for  $x, y, z$  and adding, we obtain:

$$\sum_{cyc} \frac{\tan^2 x}{\tan^3 x + \cot x} + \sum_{cyc} \frac{\cot^2 x}{\cot^3 x + \tan x} \geq 2 \sum_{cyc} \frac{1}{\tan^2 x + \cot^2 x}$$

**Note by editor:**

Many thanks to Florică Anastase-Romania for typed solutions.