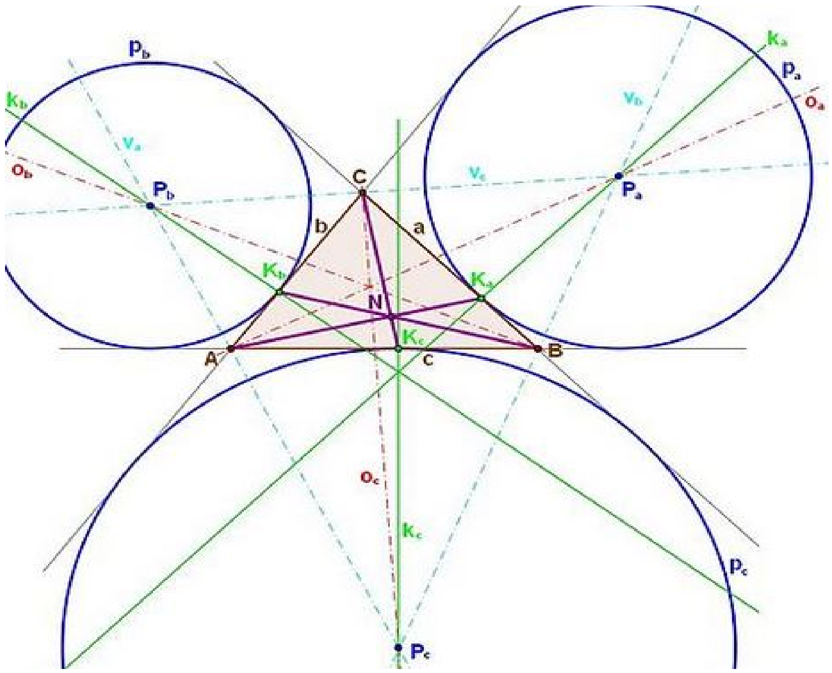

JP.357 In $\Delta ABC, N_a$ –Nagel's point. Prove that if N_a lies on incircle if and only if $s^2 + 4r^2 = 16Rr$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer, Solution 2 by Daniel Văcaru-Romania

Solution 1 by proposer

If that inscribed circle of ΔABC passes through point $N_a \Leftrightarrow IN_a = r \Leftrightarrow IN_a^2 = r^2; (1)$

$$\overrightarrow{IN_a} = \frac{(s-a)\overrightarrow{IA} + (s-b)\overrightarrow{IB} + (s-c)\overrightarrow{IC}}{s} \Leftrightarrow$$

$$IN^2 = \frac{\sum(s-a)^2 IA^2 + \sum bc(s-a)\overrightarrow{IA}\cdot\overrightarrow{IB}}{s^2}; (2)$$

$$\sum(s-a)^2 IA^2 = \sum(s-a)^2(bc-4Rr) = \sum bc(s-a)^2 - 4Rr\sum(s-a)^2 =$$

$$= abc\sum\frac{(s-a)^2}{a} - 4Rr\sum(s-a)^2 = 4Rrs\cdot\frac{s(s^2+r^2-12Rr)}{4Rr} - 4Rr$$

$$s^2 - 2r^2 - 8Rr = s^2(s^2+r^2-12Rr) - 4Rr(s^2-2r^2-8Rr) =$$

$$s^2(s^2+r^2-12Rr) - 4Rrs^2 + 4Rr(2r^2+8Rr) =$$

$$= s^2(s^2 + r^2 - 16Rr) + 8Rr^2(4R + r); (3)$$

$$\sum(s-a)(s-b)\overrightarrow{IA}\cdot\overrightarrow{IB} = \sum(s-a)(s-b)\left(\frac{IA^2 + IB^2 - AB^2}{2}\right) =$$

$$= \sum(s-a)(s-b)\left(\frac{bc - 4Rr + ac - 4Rr - c^2}{2}\right) =$$

$$= \sum(s-a)(s-b)\left(\frac{2c(s-c) - 8Rr}{2}\right) =$$

$$= \sum(s-a)(s-b)(c(s-c) - 4Rr) =$$

$$= (s-a)(s-b)(s-c)\cdot 2s - 4Rr\sum(s-a)(s-b) =$$

$$= 2s^2r^2 - 4Rr\big(r(4R + r)\big) = 2s^2r^2 - 4Rr^2(4R + r); (4)$$

From (2),(3),(4) it follows that:

$$IN^2 = s^2 + r^2 - 16Rr + \frac{8Rr^2(4R + r)}{s^2} + 4r^2 - \frac{8Rr^2(4R + r)}{s^2} = s^2 + 5r^2 - 16Rr; (5)$$

From (1),(5) it follows that: $s^2 + 4r^2 = 16Rr$.

Solution 2 by Daniel Văcaru-Romania

We know that: $N_aI = 3IG \Rightarrow 3IG = r \Rightarrow r^2 = 9GI^2 \Rightarrow r^2 = s^2 + 5r^2 - 16Rr \Rightarrow$

$16Rr = s^2 + 4r^2$. **Or, else:**

$$N_aM^2 = \frac{s-a}{s}MA^2 + \frac{s-b}{s}MB^2 + \frac{s-c}{s}MC^2 + 4r^2 - 4Rr; (*)$$

We obtain $N_aI^2 = \frac{s-a}{s}IA^2 + \frac{s-b}{s}IB^2 + \frac{s-c}{s}IC^2 + 4r^2 - 4Rr$ **and**

$$AI = \frac{bc}{s}\cos\frac{A}{2}, AI^2 = \frac{b^2c^2}{s^2}\cdot\frac{s(s-a)}{s^2} \Rightarrow \frac{s-a}{s}AI^2 = \frac{bc(s-a)^2}{s^2}$$

$$\sum_{cyc}\frac{s-a}{s}IA^2 = \sum_{cyc}\frac{bc(s-a)^2}{s^2} = \frac{s^2\sum ab - 6abcs + abc(a+b+c)}{s^2} =$$

$$= \frac{s^2\sum ab - 4abcs}{s^2} = \frac{s^2(s^2 + r^2 + 4Rr) - 16Rrs^2}{s^2} = s^2 + r^2 - 12Rr$$

Again in $(*)$**, we obtain**

$$N_aI^2 = s^2 + r^2 - 12Rr + 4R^2 - 4Rr = s^2 + 5r^2 - 16Rr.$$

But $N_aI = r \Rightarrow r^2 = s^2 + 5r^2 - 16Rr$.

Therefore, $s^2 + 4r^2 = 16Rr$.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.