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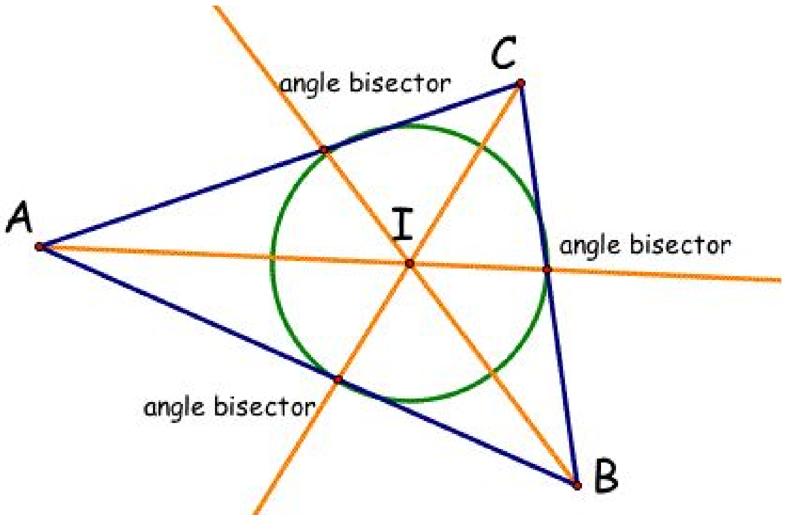

JP.356 In $\Delta ABC, I$ −incenter and R_a, R_b, R_c −circumradii in $\Delta IBC, \Delta IAB, \Delta IAC$. Prove that:

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\ge 1$$

Proposed by Marian Ursărescu-Romania

Solution 1 by proposer, Solution 2 by Daniel Văcaru-Romania

Solution 1 by proposer

$$\mu(\widehat{BIC})=\pi-\frac{B}{2}-\frac{C}{2}=\frac{\pi}{2}+\frac{A}{2}\Rightarrow$$

$$2R_a=\frac{a}{sin\left(\frac{\pi}{2}+\frac{A}{2}\right)}=\frac{a}{cos\frac{A}{2}}=\frac{2RsinA}{cos\frac{A}{2}}=4Rsin\frac{A}{2}\Rightarrow R_a=2Rsin\frac{A}{2}(and\ analogs)$$

$$\frac{R_a}{a}=\frac{2Rsin\frac{A}{2}}{2RsinA}=\frac{sin\frac{A}{2}}{sinA}=\frac{1}{2cos\frac{A}{2}}\Rightarrow\left(\frac{R_a}{a}\right)^2=\frac{1}{4cos^2\frac{A}{2}}$$

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2=\frac{1}{4}\sum_{cyc}\frac{1}{cos^2\frac{A}{2}}$$

But $\sum_{cyc}\frac{1}{cos^2\frac{A}{2}}=1+\left(\frac{4R+r}{s}\right)^2$ and $(4R+r)^2\ge 3s^2(Doucet)\Rightarrow$

$$\left(\frac{R_a}{a}\right)^2+\left(\frac{R_b}{b}\right)^2+\left(\frac{R_c}{c}\right)^2\geq 1$$

Solution 2 by Daniel Văcaru-Romania

BY Sine Law in ΔIBC, $a=2R_a sin(\widehat{BIC})=2R_a sin\left(\pi-\frac{B}{2}-\frac{C}{2}\right)=$

$$=2R_a sin\left(\frac{B+C}{2}\right)=2R_a cos\frac{A}{2}\Rightarrow\frac{R_a}{a}=\frac{1}{2cos\frac{A}{2}}\Rightarrow$$

$$\left(\frac{R_a}{a}\right)^2=\sum_{cyc}\frac{1}{4cos^2\frac{A}{2}}\overset{Bergstrom}{\geq}\frac{9}{4\sum cos^2\frac{A}{2}}=\frac{9}{4\left(2+\frac{r}{2R}\right)}\overset{Euler}{\geq}1$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.