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For $1 \leq a, b, c \leq \frac{2\sqrt{3}}{3}$, prove that:

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c)\leq 3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by proposer, Solution 2 by Tran Hong-Dong Thap-Vietnam

Solution 1 by proposer

First, will prove that $\sqrt{4-3a^2} \leq -3a^2+3a+1$. We have:

$$-3a^2+3a+1-\sqrt{4-3a^2} = 3a(1-a)+\frac{1-(4-3a^2)}{1+\sqrt{4-3a^2}} =$$

$$= 3a(1-a)+\frac{3a^2-3}{1+\sqrt{4-3a^2}} = 3a(1-a)+\frac{3(a-1)(a+1)}{1+\sqrt{4-3a^2}} =$$

$$= 3a(1-a)-3(1-a)\cdot\frac{a+1}{1+\sqrt{4-3a^2}} = 3a(1-a)\left(\frac{a+1}{1+\sqrt{4-3a^2}}-1\right) =$$

$$= 3a(a-1)\cdot\frac{a+1-1-\sqrt{4-3a^2}}{1+\sqrt{4-3a^2}} = 3a(a-1)\left(\frac{a-\sqrt{4-3a^2}}{1+\sqrt{4-3a^2}}\right) =$$

$$= 3a(a-1)\frac{a^2-4+3a^2}{\left(a+\sqrt{4-3a^2}\right)\left(1+\sqrt{4-3a^2}\right)} =$$

$$= 3a(a-1)\cdot 4\cdot\frac{a^2-1}{\left(a+\sqrt{4-3a^2}\right)\left(a+\sqrt{4-3a^2}\right)} = \frac{12a(a-1)^2(a+1)}{\left(a+\sqrt{4-3a^2}\right)\left(1+\sqrt{4-3a^2}\right)} \ge 0$$

Equality holds when $a = 1$. Similarly:

$$\sqrt{4-3b^2} \le -3b^2+3b+1, \sqrt{4-3c^2} \le -3c^2+3c+1$$

So,

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2} \le -3(a^2+b^2+c^2)+3(a+b+c)+3$$

We know that $a^2+b^2+c^2 \ge \frac{1}{3}(a+b+c)^2$. So,

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2} \le -(a+b+c)^2+3(a+b+c)+3, \text{ namely}$$

$$\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2}+(a+b+c)^2-3(a+b+c) \le 3$$

Equality holds when $a = b = c = 1$.

Solution 2 by Tran Hong-Dong Thap-Vietnam

Because: $1 \le a, b, c \le \frac{2\sqrt{3}}{3} \Rightarrow 3 \le a+b+c \le 2\sqrt{3}, ; 4-3a^2, 4-3b^2, 4-3c^2 \ge 0$

Now, $\sqrt{4-3a^2}+\sqrt{4-3b^2}+\sqrt{4-3c^2} \overset{C-S}{\le} \sqrt{3[12-(a^2+b^2+c^2)]} \overset{\sum a^2 \ge \frac{(\sum a)^2}{3}}{\le}$

$$\le \sqrt{3\left(12-\left(\sum a\right)^2\right)} = \sqrt{3(12-t^2)}$$

We need to prove:

$$\sqrt{3(12-t^2)}+t^2-3t \le 3; \left(\because 3 \le t \le 2\sqrt{3}\right) \Leftrightarrow$$

$$\sqrt{3(12-t^2)} \le 3t+3-t^2 \Leftrightarrow$$

$$3(12-t^2) \le (3t+3-t^2)^2; \left(\because\ 3 \le t \le 2\sqrt{3} \Rightarrow 3t+3-t^2 \ge 6\sqrt{3}-9 > 0\right) \Leftrightarrow$$

$t^4-6t^3+6t^2+18t-27 \ge 0$, which is true because: $t \ge 3 \Rightarrow (t-3)^2 \ge 0 \Rightarrow$

$$t^2-3 \ge 6 > 0.$$

Note by editor:
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