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JP.348 If $a, b, c > 0$ then:

$$\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)\geq\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer, Solution 2 by Tan Tey Tan-China, Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 4 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 5 by Samar Das-India, Solution 6 by Tran Hong-Dong Thap-Vietnam

Solution 1 by proposer

We will prove:

$$\sum_{cyc}\frac{a^4}{b^4}\geq\sum_{cyc}\frac{a^2}{b^2}\quad (1)$$

$$\sum_{cyc}\frac{a^3}{b^3}\geq\sum_{cyc}\frac{a^2}{b^2}\quad (2)$$

$(1)\Leftrightarrow\sum_{cyc}a^8c^4\geq\sum_{cyc}a^6b^2c^4$ (by multiplying with $(abc)^4$)

$$\sum_{cyc}a^8c^4=\frac{1}{12}\sum_{cyc}12a^8c^4=\frac{1}{12}\sum_{cyc}(8a^8c^4+2a^8c^4+2a^8c^4)=$$

$$=\frac{1}{12}\sum_{cyc}(8a^8c^4+2b^8a^4+2c^8b^4)\geq$$

$$\overset{AM-GM}{\geq} \frac{1}{12}\sum_{cyc} 12\sqrt[12]{(a^8c^4)^8\cdot(b^8a^4)^2\cdot(c^8b^4)^2} = \sum_{cyc}\sqrt[12]{a^{72}b^{24}c^{48}} = \sum_{cyc} a^6b^2c^4$$

(2) $\Leftrightarrow \sum_{cyc} a^6c^3 \geq \sum_{cyc} a^5bc^3$ **(by multiplying with** $(abc)^3$**)**

$$\sum_{cyc} a^6c^3 = \frac{1}{9}\sum_{cyc} 9a^6c^3 = \frac{1}{9}\sum_{cyc}(7a^6c^3 + a^6c^3 + a^6c^3) =$$

$$= \frac{1}{9}\sum_{cyc}(7a^6c^3 + b^6a^3 + c^6b^3) \overset{AM-GM}{\geq} \frac{1}{9}\sum_{cyc} 9\sqrt[9]{(a^6c^3)^7\cdot b^6a^3\cdot c^6b^3} =$$

$$= \sum_{cyc}\sqrt[9]{a^{45}\cdot b^9\cdot c^{27}} = \sum_{cyc} a^5bc^3$$

By multiplying (1); (2):

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right) \geq \left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Equality holds for $a = b = c$**.**

Solution 2 by Tan Tey Tan-China

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right) \geq \left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(3\cdot\sqrt[3]{\prod_{cyc}\frac{a^3}{b^3}}\right) = \left(\sum_{cyc}\frac{a^4}{b^4}\right)\cdot 3 =$$

$$= \left(\sum_{cyc}\frac{a^4}{b^4}\right)(1+1+1) \geq \left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\left(\frac{a}{b}\right)^4 \overset{CBS}{\geq} \frac{1}{3}\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^3 \overset{AM-GM}{\geq} 3\prod_{cyc}\left(\frac{a}{b}\right) = 3$$

Therefore,

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\ge\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 4 by SanongHuayrerai-NakonPathom-Thailand

$$\left(\frac{a^4}{b^4}+\frac{b^4}{c^4}+\frac{c^4}{a^4}\right)\left(\frac{a^3}{b^3}+\frac{b^3}{c^3}+\frac{c^3}{a^3}\right)\ge\frac{\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}{9}\ge$$

$$\ge\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)=\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)^2$$

Solution 5 by Samar Das-India

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\ge 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^4}=3;(1);\sum_{cyc}\left(\frac{a}{b}\right)^4\ge 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^3}=3;(2)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^2\ge 3\sqrt[3]{\prod_{cyc}\left(\frac{a}{b}\right)^2}=3;(3)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4=\sum_{cyc}\left(\frac{a}{b}\right)^{2+2}\ge\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)^2=\frac{1}{3}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2;(4)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^3=\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right);(5);\ \sum_{cyc}\frac{a}{b}\ge 3;(6)$$

$$\sum_{cyc}\left(\frac{a}{b}\right)^4\cdot\sum_{cyc}\left(\frac{a}{b}\right)^3-\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\ge\frac{1}{3}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\cdot\frac{1}{3}\sum_{cyc}\left(\frac{a}{b}\right)-\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2=$$

$$=\frac{1}{9}\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\right)^2\left[\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)-9\right]=\frac{1}{9}\sum_{cyc}\left(\frac{a}{b}\right)^2\left(\sum_{cyc}\left(\frac{a}{b}\right)^2\cdot\sum_{cyc}\left(\frac{a}{b}\right)-9\right)$$

Therefore,

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\ge\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Solution 6 by Tran Hong-Dong Thap-Vietnam

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Let $x=\frac{a}{b}>0, y=\frac{b}{c}>0, z=\frac{c}{a}>0 \Rightarrow xyz=1$

$$x^4+y^4+z^4=x^2\cdot x^2+y^2\cdot y^2+z^2\cdot z^2 \overset{Chebyshev's}{\geq} \frac{1}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\cdot \sqrt[3]{(xyz)^2}\cdot \sum_{cyc}x^2=\sum_{cyc}x^2;(1)$$

$$x^3+y^3+z^3=x\cdot x^2+y\cdot y^2+z\cdot z^2 \overset{Chebyshev's}{\geq} \frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\cdot 3\cdot \sum_{cyc}x^2=\sum_{cyc}x^2;(2)$$

From (1),(2) it follows that:

$$\left(\sum_{cyc}x^4\right)\left(\sum_{cyc}x^3\right)\geq\left(\sum_{cyc}x^2\right)^2$$

Therfore,

$$\left(\sum_{cyc}\frac{a^4}{b^4}\right)\left(\sum_{cyc}\frac{a^3}{b^3}\right)\geq\left(\sum_{cyc}\frac{a^2}{b^2}\right)^2$$

Equality holds for $a=b=c$.

Note by editor:
Many thanks to FloricăAnastase-Romania-for typed solutions.