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JP.347 Let a, b, c be non-negative real numbers, no two of which are zero. Prove that

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by proposer, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 3 by Tran Hong-Dong Thap-Vietnam

Solution 1 by proposer

We rewrite the inequality as

$$\sum_{cyc}\frac{2(a^2+ab+bc+ca)}{b+c}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

Or equivalent to

$$\sum_{cyc}\left(\frac{2(a^2+bc)}{b+c}+2a\right)>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c}-2(a+b+c)$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{[(a+b+c)^2-(ab+bc+ca)]^2}{2(ab+bc+ca)(a+b+c)}$$

$$\sum_{cyc}\frac{2(a^2+bc)}{b+c}>\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2(ab+bc+ca)(a+b+c)}$$

Now, we use the Cauchy-Schwarz inequality to obtain

$$\sum_{cyc}\frac{a^2+bc}{b+c}=\sum_{cyc}\frac{(a^2+bc)^2}{(a^2+bc)(b+c)}\geq\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{\sum(a^2+bc)(b+c)}=$$

$$=\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2\sum bc(b+c)}\geq\frac{(a^2+b^2+c^2+ab+bc+ca)^2}{2(ab+bc+ca)(a+b+c)}$$

Note that equality can't happen. The proof is completed.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{2(a+b)(a+c)}{b+c}+\frac{2(b+c)(b+a)}{c+a}+\frac{2(c+a)(c+b)}{a+b}$$

$$>\frac{(a+b+c)^3}{ab+bc+ca}+\frac{ab+bc+ca}{a+b+c};(*)$$

$$(*)\Leftrightarrow 2\sum_{cyc}\frac{a^2}{b+c}+2\sum_{cyc}a+2\sum_{cyc}\frac{bc}{b+c}>\frac{(\sum a)^3}{\sum ab}+\frac{\sum ab}{\sum a}\Leftrightarrow$$

$$2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{a^2}{b+c}\right)+2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{bc}{b+c}\right)$$

$$>\left(\sum_{cyc}a^3\right)+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$2\sum_{cyc}\frac{a^3b}{b+c}+2\sum_{cyc}\frac{a^3c}{b+c}+2abc\sum_{cyc}\frac{a}{b+c}+2\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+2abc\sum_{cyc}\frac{b}{b+c}$$

$$+2abc\sum_{cyc}\frac{c}{b+c}+2\sum_{cyc}\frac{(bc)^2}{b+c}>\left(\sum_{cyc}a\right)^3+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$2\sum_{cyc}a^3+2abc\sum_{cyc}\frac{a}{b+c}+2\sum_{cyc}a^2b+6abc+6abc+2\sum_{cyc}\frac{(bc)^2}{b+c}$$

$$>\sum_{cyc}a^3+3\sum_{cyc}a^2b+6abc+\frac{(\sum ab)^2}{\sum a}\Leftrightarrow$$

$$\left(\sum_{cyc} a^3 + 3abc\right) + 2\sum_{cyc}\frac{(bc)^2}{b+c} + 2abc\sum_{cyc}\frac{a}{b+c} + 3abc > \sum_{cyc} a^2b + \frac{(\sum ab)^2}{\sum a};(1)$$

which is true, because $\sum a^3 + 3abc \geq \sum a^2b(Schur's)$ **and** $2\sum\frac{b^2c^2}{b+c} \geq \frac{(\sum bc)^2}{\sum a}(BCS)$.

Solution 3 by Tran Hong-Dong Thap-Vietnam

We have: $a+b>0, b+c>0, c+a>0$.

$$\sum_{cyc}\frac{2(a+b)(b+c)}{b+c} = 2\sum_{cyc}\frac{\left((a+b)(b+c)\right)^2}{(a+b)(b+c)(c+a)} = 2\frac{\sum\left((a+b)(b+c)\right)^2}{(a+b)(b+c)(c+a)} =$$

$$= \frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)-abc} > \frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)}$$

We need to prove that:

$$\frac{2\sum\left((a+b)(a+c)\right)^2}{(a+b+c)(ab+bc+ca)} > \frac{(a+b+c)^3}{ab+bc+ca} + \frac{ab+bc+ca}{a+b+c} \Leftrightarrow$$

$$2\sum_{cyc}\left((a+b)(a+c)\right)^2 > (a+b+c)^4 + (ab+bc+ca)^2 \Leftrightarrow$$

$$2[(a^2+ab+bc+ca)^2 + (b^2+ab+bc+ca)^2 + (c^2+ab+bc+ca)^2]$$
$$> (a+b+c)^4 + (ab+bc+ca)^2 \Leftrightarrow$$

$$2[a^4+b^4+c^4+2(a^2+b^2+c^2)(ab+bc+ca)+3(ab+bc+ca)^2]$$
$$> \left(a^2+b^2+c^2+2(ab+bc+ca)\right)^2 + (ab+bc+ca)^2 \Leftrightarrow$$

$$2\left[\sum_{cyc}a^4 + 2\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right) + 3\left(\sum_{cyc}ab\right)^2\right] >$$

$$> \left(\sum_{cyc}a^2\right)^2 + 4\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}ab\right) + 5\left(\sum_{cyc}ab\right)^2 \Leftrightarrow$$

$$2\sum_{cyc}a^4 + \left(\sum_{cyc}ab\right)^2 > \left(\sum_{cyc}a^2\right)^2 \Leftrightarrow \sum_{cyc}a^4 + \left(\sum_{cyc}ab\right)^2 > 2\sum_{cyc}a^2b^2 \Leftrightarrow$$

$$\sum_{cyc} a^4 + 2abc(a+b+c) > \sum_{cyc} a^2b^2$$

Which is true, because: $x^2 + y^2 + z^2 \geq xy + yz + zx$**. Hence,**

$$\sum_{cyc}(a^2)^2 \geq \sum_{cyc} a^2b^2 \Rightarrow \sum_{cyc} a^4 + 2abc(a+b+c) > \sum_{cyc} a^2b^2$$

Note by editor:
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