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JP.346 Find all values of k such that the following inequality:

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b)$$

holds for all positive real numbers a,b.

Proposed by Nguyen Viet Hung-Hanoi-Vietnam

Solution 1 by Alex Szoros-Romania, Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 3 by proposer

Solution 1 by Alex Szoros-Romania

$$\frac{a^2}{b}+\frac{b^2}{a}+\frac{kab}{a+b}\geq\left(1+\frac{k}{4}\right)(a+b);(*)\Leftrightarrow\frac{a^2}{b}+\frac{b^2}{a}-(a+b)+k\left(\frac{ab}{a+b}-\frac{a+b}{4}\right)\geq 0$$

Let be the function $f:\mathbb{R}\to\mathbb{R}, f(k)=\left(\frac{ab}{a+b}-\frac{a+b}{4}\right)k+\frac{a^2}{b}+\frac{b^2}{a}-(a+b)$

$$f'(k)=\frac{ab}{a+b}-\frac{a+b}{4}=\frac{4ab-(a+b)^2}{4(a+b)}\Rightarrow f'(k)=-\frac{(a-b)^2}{4(a+b)}\leq 0,\forall a,b>0\Rightarrow$$

f −decreasing; (1)

But $f(16)=\frac{a^2}{b}+\frac{b^2}{a}-(a+b)-\frac{16(a-b)^2}{4(a+b)}=\frac{(a+b)(a-b)^2}{ab}-\frac{4(a-b)^2}{a+b}$

$$f(16)=(a-b)^2\left(\frac{a+b}{ab}-\frac{4}{a+b}\right)=(a-b)^2\left[\frac{(a+b)^2-4ab}{ab(a+b)}\right]=\frac{(a-b)^4}{ab(a+b)}\geq 0,$$

$\forall a, b > 0 \Rightarrow f(16) \geq 0$; (2)

From (1),(2) it follows that: $k \leq 16 \Rightarrow f(k) \geq f(16) \geq 0$, so $f(k) \geq 0, \forall k \leq 16$.

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{a^2}{b} + \frac{b^2}{a} + \frac{kab}{a+b} \geq \left(1 + \frac{k}{4}\right)(a+b) \Leftrightarrow \frac{a^2}{b} + \frac{b^2}{a} - (a+b) + k\left(\frac{ab}{a+b} - \frac{a+b}{4}\right) \geq 0$$

$$\Leftrightarrow \frac{(a-b)^2(a+b)}{ab} \geq \frac{k(a-b)^2}{4(a+b)}$$

For $a = b \Rightarrow (*)$ holds for all $k \Rightarrow (*) \Leftrightarrow (a+b)^2 \geq \frac{kab}{4}$

If $k \leq 0 \Rightarrow (*)$ holds for all $a, b > 0$. Let $k > 0$ and let $x = \frac{a}{b}$, then

$$(*) \Leftrightarrow x^2 + \left(2 - \frac{k}{4}\right)x + 1 \geq 0, \text{ with } \Delta = \left(2 - \frac{k}{4}\right)^2 - 4 = \frac{k(k-16)}{16}$$

$k \leq 0 \Rightarrow \Delta \leq 0 \Rightarrow (*)$ holds for all $a, b > 0$.

If $k \leq 16$, and if $0 < x < \frac{1}{2}\left(-2 + \frac{k}{4} + \sqrt{\Delta}\right) > 0 \Rightarrow x^2 + \left(2 - \frac{k}{4}\right)x + 1 < 0$.

Therefore, $k \leq 16$.

Solution 3 by proposer

The inequality is equivalent to

$$\frac{a^2}{b} + \frac{b^2}{c} - (a+b) + \frac{kab}{a+b} - \frac{k(a+b)}{4} \geq 0$$

$$\frac{a^3 + b^3 - ab(a+b)}{ab} - \frac{k[(a+b)^2 - 4ab]}{4(a+b)} \geq 0$$

$$\left(\frac{a+b}{ab} - \frac{k}{4(a+b)}\right)(a-b)^2 \geq 0$$

$$\frac{a+b}{ab} \geq \frac{k}{4(a+b)}$$

$$\frac{4(a+b)^2}{ab} \geq k$$

This is true for all positive real numbers a, b if and only if

$$\min_{a,b>0} \frac{4(a+b)^2}{ab} \geq k$$

Because $4(a+b)^2 \geq 16ab$ and the equality happens when $a = b$, it follows that

$$\min_{a,b>0}\frac{4(a+b)^2}{ab}=16$$

So, $k\le 16$.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.