REFINEMENT FOR SOME GEOMETRICAL INEQUALITIES

MIHÁLY BENCZE, DANIEL SITARU - ROMANIA

Abstract. In this paper we present some refinement for the classical geometrical inequality.

Theorem 1.
If $x, y, z > 0$ then

$$\sum xy \le \sum F(x^2, y^2) \le \sum x^2$$

$$\text{where } F(x,y) = \frac{1}{e}\Big(\frac{y^y}{x^x}\Big)^{\frac{1}{y-x}}$$

Proof. Using the Hadamard-Hermite inequality

$$f\Big(\frac{x+y}{2}\Big) \le \frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2}$$

$(x, y > 0)$ for the convex function $f(t) = -\ln t$. We obtain:

$$-\ln\Big(\frac{x+y}{2}\Big) \le -\frac{(y\ln y - y) - (x\ln x - x)}{y-x} \le \frac{-\ln x - \ln y}{2} \text{ or}$$

$$\sqrt{xy} \le \frac{1}{e}\Big(\frac{y^y}{x^x}\Big)^{\frac{1}{y-x}} = F(x,y) \le \frac{x+y}{2}$$

or $xy \le F(x^2, y^2) \le \frac{x^2+y^2}{2}$ and

$$\sum xy \le \sum F(x^2, y^2) \le \sum x^2$$

□

Corollary 1.1. In all ΔABC holds:

$$1)\ s^2 + r^2 + 4Rr \le \sum F(a^2, b^2) \le 2(s^2 - r^2 - 4Rr)$$

a refinement of the inequality $s^2 \ge 3r(4R + r)$.

$$2)\ r(4R + r) \le \sum F((s-a)^2, (s-b)^2) \le s^2 - 2r^2 - 8Rr$$

a refinement of the inequality $s^2 \ge 3r(4R + r)$.

$$3)\ s^2 \le \sum F(r_a^2, r_b^2) \le (4R + r)^2 - 2s^2$$

a refinement of the inequality $4R + r \ge s\sqrt{3}$

$$4)\ \frac{s^2 + r^2 - 8Rr}{16R^2} \le \sum F\Big(\sin^4\frac{A}{2}, \sin^4\frac{B}{2}\Big) \le \frac{8R^2 + r^2 - s^2}{8R^2}$$

a refinement of the inequality $4R + r \ge s\sqrt{3}$.

$$5)\ \frac{s^2 + (4R + r)^2}{16R^2} \le \sum F\Big(\cos^4\frac{A}{2}, \cos^4\frac{B}{2}\Big) \le \frac{(4R + r)^2 - s^2}{8R^2}$$

a refinement of the inequality $4R + r \geq s\sqrt{3}$.
Theorem 2. If $x, y, z > 0$ then

$$8xyz \leq G(x, y, z) \leq (x + y)(y + z)(z + x)$$

where $G(x, y, z) = 8F(x, y)F(y, z)F(z, x)$

Proof. We multiply the inequality:

$$\sqrt{xy} \leq F(x, y) \leq \frac{x + y}{2}$$

□

Corollary 2.1. In all ΔABC the inequalities hold:

$$1)\ 32sRr \leq G(a, b, c) \leq 2s(s^2 + r^2 + 2Rr)$$

a refinement of the inequality $s^2 + r^2 \geq 14Rr$.

$$2)\ 8sr^2 \leq G(s - a, s - b, s - c) \leq 4sRr$$

a refinement of Euler's $R \geq 2r$ inequality.

$$3)\ 8s^2 r \leq G(r_a, r_b, r_c) \leq 4s^2 R$$

a refinement of Euler's $R \geq 2r$ inequality.

$$4)\ \frac{r^2}{2R^2} \leq G\left(\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}\right) \leq \frac{(2R - r)(s^2 + r^2 - 8Rr) - 2Rr^2}{32R^3}$$

a refinement of the inequality $18Rr^2 \leq (2R - r)(s^2 + r^2 - 8Rr)$.

$$5)\ \frac{s^2}{2R^2} \leq G\left(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2}\right) \leq \frac{(4R + r)^3 + s^2(2R + r)}{32R^3}$$

a refinement of the inequality $s^2(14R - r) \leq (4R + r)^3$.

References

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Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania
Email address: dansitaru63@yahoo.com