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Show that the last two digits of the followings

$$9^9, 9^{9^9}, 9^{9^{9^9}}, 9^{9^{9^{9^9}}}, 9^{9^{9^{9^{9^9}}}} \text{ is always } 89.$$

In general prove that the last two digits of $9 \uparrow\uparrow n = \underbrace{9^{9^{\cdot^{\cdot^{9}}}}}_{n\geq 2}$ **is 89.**

Proposed by Naren Bhandari-Bajura-Nepal

Solution by Surjeet Singhania-India

Define a sequence $x_n = 9^{x_{n-1}}, \forall n \in N$ and $x_0 = 9$.

Claim: Every element of sequence have last two digits are 89.

Let's check for $n = 1, x_1 \equiv 1\ (mod 4)$ and $x_1 \equiv 4^3(mod 25) \equiv 14(mod 25)$.

Let's solve these congruence. Since $x_1 \equiv 1(mod\ 4) \to x_1 = 4k_1 + 1$ also

$x_1 \equiv 14(mod\ 25) \to 4k_1 \equiv 13(mod 25) \to k_1 \equiv 22(mod 25) \to k_1 = 25k_2 + 22$.

Put the value of k_1 in $x_1, x_1 = 100k_2 + 89 \to k_1 \equiv 89(mod 100)$.

On he hypothesis true for $n = 1$. Assume it is true for $n = k, x_k \equiv 89(mod 100) \to$

$x_k = 100m + 89, m \in Z$. Now, we shall prove statement for $n = k + 1$.

$x_{k+1} \equiv 9^{x_k} = 9^{100m+89}, x_{k+1} \equiv 1(mod 4)$. Now, we have to mod 25 for the number.

We know that $\Phi(25) = 20$ and $100m + 89 \equiv 9(mod 20) \to$

$x_{k+1} \equiv 9^9(mod 25)(Euler\ Theorem)$.

$x_{k+1} \equiv 9^9 (mod\ 25) \equiv 14 (mod 25)$. **Since** $x_{k+1} \equiv 1 (mod 4)$.

For 6^{th} **line of our solution** $x_{k+1} \equiv 89 (mod 100)$.

Conclusion. In general prove that the last two digits of $9 \uparrow\uparrow n = \underbrace{9^{9^{\cdot^{\cdot^{9}}}}}_{n \geq 2}$ **is** 89.

Note by editor:
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