NEW LIMITS OF LALESCU TYPE WITH FIBONACCI AND LUCAS SEQUENCES

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Abstract. In this paper we present some certain Lalescu type limits with Fibonacci and Lucas numbers related to the golden ratio result.

Fibonacci sequence: $(F_n)_{n\geq 0}, F_0=0, F_1=1, F_{n+2}=F_{n+1}+F_n, \forall n\in\mathbb{N}$.
Lucas sequence: $(L_n)_{n\geq 0}, L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$.

Theorem 1.

If $(a_n)_{n\geq 1}$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+^*, (L_n)_{n\geq 0}: L_0=2, L_1=1, L_{n+2}=L_{n+1}+L_n$

$$\text{and } m\in\mathbb{N}^*, \text{ then } \lim_{n\to\infty}\left(\sqrt[m(n+1)]{a_{n+1}L_{n+1}}-\sqrt[mn]{a_nL_n}\right)\cdot n^{\frac{m-1}{m}}=\frac{1}{m}\left(\frac{a\alpha}{e}\right)^{\frac{1}{m}}$$

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_nL_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{a_nL_n}{n^n}}=\lim_{n\to\infty}\frac{a_{n+1}L_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{a_nL_n}=$$

$$=\lim_{n\to\infty}\left(\frac{a_{n+1}}{na_n}\left(\frac{n}{n+1}\right)^{n+1}\frac{L_{n+1}}{L_n}\right)=$$

$$=a\cdot\frac{1}{e}\cdot\lim_{n\to\infty}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^n+\beta^n}=\frac{a\alpha}{e},\text{ where } \alpha=\frac{\sqrt{5}+1}{2},\beta=\frac{1-\sqrt{5}}{2},L_n=\alpha^n+\beta^n$$

We denote $u_n=\frac{\sqrt[m(n+1)]{a_{n+1}L_{n+1}}}{\sqrt[mn]{a_nL_n}}$ and we have $\lim\limits_{n\to\infty}u_n=1$, so $\lim\limits_{n\to\infty}\frac{u_n-1}{\ln u_n}=1$ and

$$\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\left(\frac{a_{n+1}L_{n+1}}{a_nL_n}\cdot\frac{1}{\sqrt[n+1]{a_{n+1}L_{n+1}}}\right)^{\frac{1}{m}}=\lim_{n\to\infty}\left(\frac{a_{n+1}}{na_n}\cdot\frac{L_{n+1}}{L_n}\cdot\frac{n+1}{\sqrt[n+1]{a_{n+1}L_{n+1}}}\cdot\frac{n}{n+1}\right)^{\frac{1}{m}}=$$

$$=\left(a\cdot\alpha\cdot\frac{e}{a\alpha}\cdot 1\right)^{\frac{1}{m}}=e^{\frac{1}{m}}.$$

Hence $\lim\limits_{n\to\infty}\left(\sqrt[m(n+1)]{a_{n+1}L_{n+1}}-\sqrt[mn]{a_nL_n}\right)n^{\frac{m-1}{m}}=\lim\limits_{n\to\infty}\left(\sqrt[mn]{a_nL_n}\cdot n^{\frac{m-1}{m}}\cdot(u_n-1)\right)=$

$$=\lim_{n\to\infty}\left(\sqrt[mn]{a_nL_n}\cdot n^{\frac{m-1}{m}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n\right)=\lim_{n\to\infty}\left(\frac{\sqrt[mn]{a_nL_n}}{n^{\frac{1}{m}}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=$$

$$=\lim_{n\to\infty}\left(\left(\frac{\sqrt[n]{a_nL_n}}{n}\right)^{\frac{1}{m}}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)=\left(\frac{a\alpha}{e}\right)^{\frac{1}{m}}\cdot 1\cdot\ln e^{\frac{1}{m}}=\frac{1}{m}\left(\frac{a\alpha}{e}\right)^{\frac{1}{m}}$$

□

Theorem 2. If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^r a_n}=a\in\mathbb{R}_+^*$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n^{s+1}b_n}=b\in\mathbb{R}_+^*$ where $r,s\in\mathbb{R}_+$ then

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}=\frac{ab\alpha s}{e^{r+s+1}}$$

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{rn}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{r(n+1)}}\cdot\frac{n^{rn}}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}\left(\frac{n}{n+1}\right)^{r(n+1)}=\frac{a}{e^r}$$

and analogously $\lim\limits_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{s+1}}=\frac{b}{e^{s+1}}$. Also we have $\lim\limits_{n\to\infty}\sqrt[n]{F_n}=\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}=$

$$=\lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n}=\alpha,\text{ where }\alpha=\frac{\sqrt{5}+1}{2},\beta=\frac{1-\sqrt{5}}{2},F_n=\frac{1}{\sqrt{5}}(\alpha^n-\beta^n)$$

We denote $u_n=\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}}\cdot\frac{(n+1)^{n+s}}{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}$ and we have $\lim\limits_{n\to\infty}u_n=1$, so

$$\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1\text{ and }\lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\cdot\frac{F_{n+1}}{F_n}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n+1]{F_{n+1}}}\cdot\left(\frac{n+1}{n}\right)^{n(r+s)}=$$

$$=e^{r+s}\cdot\alpha\cdot\frac{1}{\alpha}\cdot\lim_{n\to\infty}\frac{a_n n^r}{a_{n+1}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\left(\frac{n+1}{n}\right)^r=e^{r+s}\cdot\frac{1}{a}\cdot\frac{a}{e^r}=e^s$$

$$\text{Hence }\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{F_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{F_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}=$$

$$=\lim_{n\to\infty}\left(\frac{\sqrt[n]{b_n}}{n^{s+1}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\sqrt[n]{F_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\cdot\left(\frac{n}{n+1}\right)^s\right)=\frac{b}{e^{s+1}}\cdot\frac{a}{e^r}\cdot\alpha\cdot 1\cdot\ln e^s=\frac{ab\alpha s}{e^{r+s+1}}.$$

□

Theorem 3. If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^r a_n}=a\in\mathbb{R}_+^*$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n^{s+1}b_n}=b\in\mathbb{R}_+^*$, where $r,s\in\mathbb{R}_+$, then

$$\lim_{n\to\infty}\left(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{L_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{L_n}}{(n+1)^{r+s}}\right)\sqrt[n]{b_n}=\frac{ab\alpha s}{e^{r+s+1}},$$

where $(L_n)_{n\geq 0}$ is Lucas sequence i.e. $(L_n)_{n\geq 0}, L_0=2, L_1=1$,
$L_{n+2}=L_{n+1}+L_n, \forall n\in\mathbb{N}$.

Proof. We have:

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n^r}=\lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^{rn}}}=\lim_{n\to\infty}\frac{a_{n+1}}{(n+1)^{r(n+1)}}\cdot\frac{n^{rn}}{a_n}=\lim_{n\to\infty}\frac{a_{n+1}}{n^r a_n}\left(\frac{n}{n+1}\right)^{r(n+1)}=\frac{a}{e^r}$$

and analogously $\lim\limits_{n\to\infty}\frac{\sqrt[n]{b_n}}{n^{s+1}}=\frac{b}{e^{s+1}}$. Also we have $\lim\limits_{n\to\infty}\sqrt[n]{L_n}=\lim\limits_{n\to\infty}\frac{L_{n+1}}{L_n}=$

$$=\lim_{n\to\infty}\frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^n+\beta^n}=\alpha,\text{ where }\alpha=\frac{\sqrt{5}+1}{2},\beta=\frac{1-\sqrt{5}}{2},L_n=\alpha^n+\beta^n.$$

We denote $u_n=\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{L_{n+1}}}{n^{r+s}}\cdot\frac{(n+1)^{n+s}}{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{L_n}}$ and we have $\lim\limits_{n\to\infty}u_n=1$, so

$$\lim_{n\to\infty}\frac{u_n-1}{\ln u_n}=1 \text{ and } \lim_{n\to\infty}u_n^n=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\cdot\frac{L_{n+1}}{L_n}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n+1]{L_{n+1}}}\cdot\Big(\frac{n+1}{n}\Big)^{n(r+s)}=$$

$$=e^{r+s}\cdot\alpha\cdot\frac{1}{\alpha}\cdot\lim_{n\to\infty}\frac{a_n n^r}{a_{n+1}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\Big(\frac{n+1}{n}\Big)^r=e^{r+s}\cdot\frac{1}{a}\cdot\frac{a}{e^r}=e^s$$

$$\text{Hence }\lim_{n\to\infty}\Big(\frac{\sqrt[n]{a_n}\cdot\sqrt[n+1]{L_{n+1}}}{n^{r+s}}-\frac{\sqrt[n+1]{a_{n+1}}\cdot\sqrt[n]{L_n}}{(n+1)^{r+s}}\Big)\sqrt[n]{b_n}=$$

$$=\lim_{n\to\infty}\Big(\frac{\sqrt[n]{b_n}}{n^{s+1}}\cdot\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^r}\cdot\sqrt[n]{L_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\cdot\Big(\frac{n}{n+1}\Big)^s\Big)=\frac{b}{e^{s+1}}\cdot\frac{a}{e^r}\cdot\alpha\cdot 1\cdot\ln e^s=\frac{ab\alpha s}{e^{r+s+1}}.$$

□

Theroem 4.

$$\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{F_n}{(x+1)F_{n+1}}}-(\Gamma(x+1))^{\frac{F_n}{xF_{n+1}}}\Big)x^{\frac{F_{n-1}}{F_{n+1}}}\Big)=\frac{1}{\alpha\cdot e^{\frac{1}{\alpha}}}.$$

Proof.

We denote $u_n=\dfrac{F_n}{F_{n+1}}$, we have $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\dfrac{\alpha^n-\beta^n}{\alpha^{n+1}-\beta^{n+1}}=\dfrac{1}{\alpha}$, where

$\alpha=\dfrac{\sqrt{5}+1}{2},\beta=\dfrac{1-\sqrt{5}}{2},F_n=\dfrac{1}{\sqrt{5}}(\alpha^n-\beta^n)$. Also we have

$$\lim_{n\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$$

We denote $v(x)=\dfrac{(\Gamma(x+2))^{\frac{F_n}{(x+1)F_{n+1}}}}{(\Gamma(x+1))^{\frac{F_n}{xF_{n+1}}}}=\left(\dfrac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{u_n}$, we have

$$\lim_{n\to\infty}v(x)=1,\text{ so }\lim_{n\to\infty}\frac{v(x)-1}{\ln v(x)}=1\text{ and}$$

$$\lim_{n\to\infty}(v(x))^x=\lim_{n\to\infty}\Big(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\Big)^{u_n}=\lim_{n\to\infty}\Big(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\Big)^{u_n}=e^{u_n}$$

therefore

$\lim_{n\to\infty}(\lim_{n\to\infty}(v(x))^x)=e^{\frac{1}{\alpha}}$. Hence: $\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{F_n}{(x+1)F_{n+1}}}-(\Gamma(x+1))^{\frac{F_n}{xF_{n+1}}}\Big)x^{\frac{F_{n-1}}{F_{n+1}}}=$

$$=\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\Big)x^{\frac{F_{n-1}}{F_{n+1}}}\Big)=$$

$$=\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\Big)x^{\frac{F_{n+1}-F_n}{F_{n+1}}}\Big)=$$

$$=\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+1))^{\frac{u_n}{x}}\Big)(v(x)-1)x^{1-u_n}\Big)=$$

$$=\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+1))^{\frac{u_n}{x}}\Big)\frac{v(x)-1}{\ln v(x)}x^{1-u_n}\ln v(x)\Big)=$$

$$=\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\Big)^{u_n}\frac{v(x)-1}{\ln v(x)}\ln(v(x))^x\Big)=$$

$$=\lim_{n\to\infty}\Big(\Big(\frac{1}{e}\Big)^{u_n}\cdot 1\cdot\ln e^{u_n}\Big)=\Big(\frac{1}{e}\Big)^{\frac{1}{\alpha}}\ln e^{\frac{1}{\alpha}}=\frac{1}{\alpha\cdot e^{\frac{1}{\alpha}}}$$

□

Theorem 5.

$$\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+2))^{\frac{L_n}{(x+1)L_{n+1}}}-(\Gamma(x+1))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)=\frac{1}{\alpha\cdot e^{\frac{1}{\alpha}}}$$

Proof.

We denote $u_n=\frac{L_n}{L_{n+1}}$, we have $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}}=\frac{1}{\alpha}$, where

$$\alpha=\frac{\sqrt{5}+1}{2},\beta=\frac{1-\sqrt{5}}{2},L_n=\alpha^n+\beta^n.\text{ Also we have}$$

$$\lim_{n\to\infty}\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}=\lim_{n\to\infty}\frac{(\Gamma(n+1))^{\frac{1}{n}}}{n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}.$$

We denote $v(x)=\frac{(\Gamma(x+2))^{\frac{L_n}{(x+1)L_{n+1}}}}{(\Gamma(x+1))^{\frac{L_n}{xL_{n+1}}}}=\left(\frac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\right)^{u_n}$, we have

$$\lim_{n\to\infty}v(x)=1,\text{ so }\lim_{n\to\infty}\frac{v(x)-1}{\ln v(x)}=1\text{ and}$$

$$\lim_{n\to\infty}(v(x))^x=\lim_{n\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{u_n}=\lim_{n\to\infty}\left(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\right)^{u_n}=e^{u_n}$$

therefore

$$\lim_{n\to\infty}(\lim_{n\to\infty}(v(x))^x)=e^{\frac{1}{\alpha}}.\text{ Hence:}$$

$$\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+2))^{\frac{L_n}{(x+1)L_{n+1}}}-(\Gamma(x+1))^{\frac{L_n}{xL_{n+1}}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}(\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\right)x^{\frac{L_{n-1}}{L_{n+1}}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+2))^{\frac{u_n}{x+1}}-(\Gamma(x+1))^{\frac{u_n}{x}}\right)x^{\frac{L_{n+1}-L_n}{L_{n+1}}}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+1))^{\frac{u_n}{x}}\right)(v(x)-1)x^{1-u_n}\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left((\Gamma(x+1))^{\frac{u_n}{x}}\right)\frac{v(x)-1}{\ln v(x)}x^{1-u_n}\ln v(x)\right)=$$

$$=\lim_{n\to\infty}\left(\lim_{n\to\infty}\left(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\right)^{u_n}\frac{v(x)-1}{\ln v(x)}\ln(v(x))^x\right)=$$

$$=\lim_{n\to\infty}\left(\left(\frac{1}{e}\right)^{u_n}\cdot 1\cdot\ln e^{u_n}\right)=\left(\frac{1}{e}\right)^{\frac{1}{\alpha}}\ln e^{\frac{1}{\alpha}}=\frac{1}{\alpha e^{\frac{1}{\alpha}}}$$

□

Theorem 6.

$$\lim_{n\to\infty}\left(\lim_{n\to\infty}(f(x+1))^{\frac{F_n}{(x+1)F_{n+1}}}-(f(x))^{\frac{F_n}{xF_{n+1}}}\right)x^{\frac{F_{n-1}}{F_{n+1}}}\right)=\left(\frac{a}{e}\right)^{\frac{1}{\alpha}}\frac{1}{\alpha}(1+\ln a)$$

where $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function which verify that $\lim_{n\to\infty}\frac{f(x+1)}{xf(x)}=a\in\mathbb{R}_+^*$

Proof.

We denote $u_n = \dfrac{F_n}{F_{n+1}}$, we have $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \dfrac{1}{\alpha}$, where

$$\alpha = \frac{\sqrt{5}+1}{2}, \beta = \frac{1-\sqrt{5}}{2}, F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \text{ Also we have}$$

$$\lim_{n\to\infty} \frac{(f(x))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(f(n))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{f(n)}{n^n}} = \lim_{n\to\infty} \frac{f(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{f(n)} =$$

$$= \lim_{n\to\infty} \frac{f(n+1)}{nf(n)} \Big(\frac{n}{n+1}\Big)^{n+1} = \frac{a}{e}$$

We denote $v(x) = \dfrac{(f(x+1))^{\frac{F_n}{(x+1)F_{n+1}}}}{(f(x))^{\frac{F_n}{xF_{n+1}}}} = \Big(\dfrac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\Big)^{u_n}$ we have $\lim\limits_{n\to\infty} v(x) = 1$, so

$$\lim_{n\to\infty} \frac{v(x)-1}{\ln v(x)} = 1 \text{ and } \lim_{n\to\infty} (v(x))^x = \lim_{n\to\infty} \Big(\frac{f(x+1)}{f(x)} \cdot \frac{1}{(f(x+1))^{\frac{1}{x+1}}}\Big)^{u_n} =$$

$$= \lim_{n\to\infty} \Big(\frac{x+1}{(f(x+1))^{\frac{1}{x+1}}}\Big)^{u_n} = (ae)^{u_n}$$

therefore

$$\lim_{n\to\infty} (\lim_{n\to\infty} (v(x))^x) = (ae)^{\frac{1}{\alpha}}. \text{ Hence:}$$

$$\lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{F_n}{(x+1)F_{n+1}}} - (f(x))^{\frac{F_n}{xF_{n+1}}}\Big) x^{\frac{F_{n-1}}{F_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{u_n}{x+1}} - (f(x))^{\frac{u_n}{x}}\Big) x^{\frac{F_{n-1}}{F_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{u_n}{x+1}} - (f(x))^{\frac{u_n}{x}}\Big) x^{\frac{F_{n+1}-F_n}{F_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x))^{\frac{u_n}{x}}\Big)(v(x)-1)x^{1-u_n}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x))^{\frac{u_n}{x}}\Big) \frac{v(x)-1}{\ln v(x)} x^{1-u_n} \ln v(x)\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big(\frac{(f(x))^{\frac{1}{x}}}{x}\Big)^{u_n} \frac{v(x)-1}{\ln v(x)} \ln(v(x))^x\Big) =$$

$$= \lim_{n\to\infty}\Big(\Big(\frac{a}{e}\Big)^{u_n} \cdot 1 \cdot \ln(ae)^{u_n}\Big) = \Big(\frac{a}{e}\Big)^{\frac{1}{\alpha}} \ln(ae)^{\frac{1}{\alpha}} =$$

$$= \Big(\frac{a}{e}\Big)^{\frac{1}{\alpha}} \frac{1}{\alpha}(1 + \ln a)$$

□

Theorem 7.

$$\lim_{n\to\infty}\Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{L_n}{(x+1)F_{n+1}}} - (f(x))^{\frac{L_n}{xL_{n+1}}}\Big) x^{\frac{L_{n-1}}{L_{n+1}}}\Big) = \Big(\frac{a}{e}\Big)^{\frac{1}{\alpha}} \frac{1}{\alpha}(1 + \ln a)$$

where $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a function which verify that $\lim\limits_{n\to\infty} \dfrac{f(x+1)}{xf(x)} = a \in \mathbb{R}_+^*$

Proof.

We denote $u_n = \dfrac{L_n}{L_{n+1}}$, we have $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{\alpha^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}} = \dfrac{1}{\alpha}$, where

$$\alpha = \frac{\sqrt{5}-1}{2}, \beta = \frac{1-\sqrt{5}}{2}, L_n = \alpha^n + \beta^n. \text{ Also we have}$$

$$\lim_{n\to\infty} \frac{(F(x))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(f(n))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{f(n)}{n^n}} = \lim_{n\to\infty} \frac{f(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{f(n)} =$$

$$= \lim_{n\to\infty} \frac{f(n+1)}{nf(n)} \Big(\frac{n}{n+1}\Big)^{n+1} = \frac{a}{e}$$

We denote $v(x) = \dfrac{(f(x+1))^{\frac{L_n}{(x+1)F_{n+1}}}}{(f(x))^{\frac{L_n}{xL_{n+1}}}} = \Big(\dfrac{(f(x+1))^{\frac{1}{x+1}}}{(f(x))^{\frac{1}{x}}}\Big)^{u_n}$, we have

$\lim\limits_{n\to\infty} v(x) = 1$, so $\lim\limits_{n\to\infty} \dfrac{v(x)-1}{\ln v(x)} = 1$ and $\lim\limits_{n\to\infty} (v(x))^x = \lim\limits_{n\to\infty} \Big(\dfrac{f(x+1)}{f(x)} \cdot \dfrac{1}{(f(x+1))^{\frac{1}{x+1}}}\Big)^{u_n} =$

$$= \lim_{n\to\infty} \Big(\frac{x+1}{(f(x+1))^{\frac{1}{x+1}}}\Big)^{u_n} = (ae)^{u_n}, \text{ therefore } \lim_{n\to\infty} (\lim_{n\to\infty} (v(x))^x) = (ae)^{\frac{1}{\alpha}}$$

$$\text{Hence: } \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{L_n}{(x+1)L_{n+1}}} - (f(x))^{\frac{L_n}{xL_{n+1}}}\Big) x^{\frac{L_{n-1}}{L_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{u_n}{x+1}} - (f(x))^{\frac{u_n}{x}}\Big) x^{\frac{L_{n-1}}{L_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((f(x+1))^{\frac{u_n}{x+1}} - (f(x))^{\frac{u_n}{x}}\Big) x^{\frac{L_{n+1}-L_n}{L_{n+1}}}\Big) =$$

$$= \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((f(x))^{\frac{u_n}{x}}\Big)(v(x)-1)x^{1-u_n}\Big) =$$

$$= \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((f(x))^{\frac{u_n}{x}}\Big) \frac{v(x)-1}{\ln v(x)} x^{1-u_n} \ln v(x)\Big) =$$

$$= \lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big(\frac{(f(x))^{\frac{1}{x}}}{x}\Big)^{u_n} \frac{v(x)-1}{\ln v(x)} \ln (v(x))^x\Big) =$$

$$= \lim_{n\to\infty} \Big(\Big(\frac{a}{e}\Big)^{u_n} \cdot 1 \cdot \ln(ae)^{u_n}\Big) = \Big(\frac{a}{e}\Big)^{\frac{1}{\alpha}} \ln(ae)^{\frac{1}{\alpha}} = \Big(\frac{a}{e}\Big)^{\frac{1}{\alpha}} \frac{1}{\alpha}(1+\ln a).$$

□

Theorem 8.

$$\lim_{n\to\infty} \Big(\lim_{n\to\infty} \Big((\Gamma(x+2))^{\frac{F_{n+1}^2}{(x+1)F_{2n+1}}} - (\Gamma(x+1))^{\frac{F_{n+1}^2}{xF_{2n+1}}}\Big) x^{\frac{F_n^2}{F_{2n+1}}}\Big) = \frac{1}{\alpha\sqrt{5}e^{\frac{1}{\alpha\sqrt{5}}}}$$

Proof.

We denote $u_n = \dfrac{F_{n+1}^2}{F_{2n+1}}$, we have $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{5}} \dfrac{(\alpha^{n+1} - \beta^{n+1})^2}{\alpha^{2n+1} - \beta^{2n+1}} = \dfrac{1}{\alpha\sqrt{5}}$

where $\alpha = \dfrac{\sqrt{5}+1}{2}, \beta = \dfrac{1-\sqrt{5}}{2}, F_n = \dfrac{1}{\sqrt{5}}(\alpha^n - \beta^n)$. Also, we have:

$$\lim_{n\to\infty} \frac{(\Gamma(x+1))^{\frac{1}{x}}}{x} = \lim_{n\to\infty} \frac{(\Gamma(n+1))^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

We denote $v(x) = \Big(\dfrac{(\Gamma(x+2))^{\frac{1}{x+1}}}{(\Gamma(x+1))^{\frac{1}{x}}}\Big)^{u_n}$, we have $\lim\limits_{n\to\infty} v(x) = 1$, so $\lim\limits_{n\to\infty} \dfrac{v(x)-1}{\ln v(x)} = 1$ and

$$\lim_{n\to\infty}(v(x))^x = \lim_{n\to\infty}\Big(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\Big)^{u_n} = \lim_{n\to\infty}\Big(\frac{x+1}{(\Gamma(x+2))^{\frac{1}{x+1}}}\Big)^{u_n} = e^{u_n}$$

therefore $\lim_{n\to\infty}(\lim_{n\to\infty}(v(x))^x) = e^{\frac{1}{\alpha\sqrt{5}}}$. Hence:

$$\lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{F_{n+1}^2}{(x+1)F_{2n+1}}} - (\Gamma(x+1))^{\frac{F_{n+1}^2}{xF_{2n+1}}}\Big)x^{\frac{F_n^2}{F_{2n+1}}}\Big)$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{u_n}{x+1}} - (\Gamma(x+1))^{\frac{u_n}{x}}\Big)x^{\frac{F_n^2}{F_{2n+1}}}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+2))^{\frac{u_n}{x+1}} - (\Gamma(x+1))^{\frac{u_n}{x}}\Big)x^{\frac{F_{2n+1}-F_{n+1}^2}{F_{2n+1}}}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+1))^{\frac{u_n}{x}}\Big)(v(x)-1)x^{1-u_n}\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big((\Gamma(x+1))^{\frac{u_n}{x}}\Big)\frac{v(x)-1}{\ln v(x)}x^{1-u_n}\ln v(x)\Big) =$$

$$= \lim_{n\to\infty}\Big(\lim_{n\to\infty}\Big(\frac{(\Gamma(x+1))^{\frac{1}{x}}}{x}\Big)^{u_n}\frac{v(x)-1}{\ln v(x)}\ln(v(x))^x\Big) =$$

$$= \lim_{n\to\infty}\Big(\Big(\frac{1}{e}\Big)^{u_n}\cdot 1\cdot \ln e^{u_n}\Big) = \Big(\frac{1}{e}\Big)^{\frac{1}{\alpha\sqrt{5}}}\ln e^{\frac{1}{\alpha\sqrt{5}}} = \frac{1}{\alpha\sqrt{5}e^{\frac{1}{\alpha\sqrt{5}}}}$$

□

Theorem 9. Let the sequence $(x_n)_{n\geq 0}$ given by the recurrence $(x_n)_{n\geq 0}, x_0 = 0, x_1 = 1, x_{n+2} = (2n+5)x_{n+1} - (n^2-4n+3)x_n, \forall n\in\mathbb{N}$. Then

$$\lim_{n\to\infty}(\sqrt[n+1]{L_{n+1}x_{n+1}} - \sqrt[n]{L_nx_n}) = \frac{\alpha}{e}$$

Proof. We have $x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n \Leftrightarrow$

$x_{n+2} - (n+2)x_{n+1} = (n+3)(x_{n+1} - (n+1)x_n) \Leftrightarrow y_{n+1} = (n+3)y_n$ where $y_n = x_{n+1} - (n+1)x_n, y_0 = x_1 - x_0 = 1, y_1 = x_2 - 2x_1 = 5 - 2\cdot 1 = 3$. Therefore $y_{k+1} = (k+3)y_k, \forall k\in\mathbb{N}$, so $\prod_{k=0}^n y_{k+1} = \prod_{k=0}^n(k+3)\cdot\prod_{k=0}^n y_k$ which yields succesively to $y_{n+1} = y_0\prod_{k=0}^n(k+3) = \frac{(n+3)!}{2}$ or $y_n = \frac{(n+2)!}{2}$. Then we have successively that $x_{k+1} - (k+1)x_k = y_k = \frac{(k+2)!}{2} \Leftrightarrow$

$$\frac{x_{k+1}}{(k+1)!} - \frac{x_k}{k!} = \frac{k+2}{2} = \frac{k}{2}+1 \Leftrightarrow \sum_{k=0}^n\frac{x_{k+1}}{(k+1)!} - \sum_{k=0}^n\frac{x_k}{k!} = n+1+\frac{1}{2}\sum_{k=0}^n k =$$

$$= n+1+\frac{n(n+1)}{4} = \frac{(n+1)(n+4)}{4} \Leftrightarrow \frac{x_{n+1}}{(n+1)!} = \frac{(n+1)(n+4)}{4} \Leftrightarrow$$

$$\Leftrightarrow x_{n+1} = \frac{(n+1)(n+4)}{4}(n+1)! \Leftrightarrow x_n = \frac{n(n+3)}{4}\cdot n!, \forall n\in\mathbb{N}^*.$$ We have

$$\lim_{n\to\infty}\frac{L_n}{L_{n+1}} = \lim_{n\to\infty}\frac{\alpha^n+\beta^n}{\alpha^{n+1}+\beta^{n+1}} = \frac{1}{\alpha},$$ where $\alpha = \frac{\sqrt{5}+1}{2}, \beta = \frac{1-\sqrt{5}}{2}, L_n = \alpha^n+\beta^n$

Now we obtain that $\lim_{n\to\infty}\frac{\sqrt[n]{L_nx_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{L_nx_n}{n^n}} = \lim_{n\to\infty}\frac{L_{n+1}x_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{L_nx_n} =$

$$= \lim_{n\to\infty} \frac{L_{n+1}}{L_n} \cdot \frac{x_{n+1}}{(n+1)x_n} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{\alpha}{e} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} = \frac{\alpha}{e} \lim_{n\to\infty} \frac{(n+1)(n+4)(n+1)!}{4} \cdot \frac{4}{(n+1)^2(n+3)n!} = \frac{\alpha}{e} \cdot 1 = \frac{\alpha}{e}.$$

We denote $u_n = \frac{\sqrt[n+1]{L_{n+1}x_{n+1}}}{\sqrt[n]{L_n x_n}}$ and we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$

respectively $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{L_{n+1}x_{n+1}}{L_n x_n} \cdot \frac{1}{\sqrt[n+1]{L_{n+1}x_{n+1}}} =$

$$= \lim_{n\to\infty} \frac{L_{n+1}}{L_n} \cdot \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} \cdot \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{L_{n+1}x_{n+1}}} = \alpha \cdot 1 \cdot \frac{e}{\alpha} = e.$$

Hence, $\lim_{n\to\infty} \left(\sqrt[n+1]{L_{n+1}x_{n+1}} - \sqrt[n]{L_n x_n} \right) = \lim_{n\to\infty} \frac{\sqrt[n]{L_n x_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n =$

$$= \frac{\alpha}{e} \cdot 1 \cdot \ln e = \frac{\alpha}{e}$$

□

Theorem 10. Let the sequence $(x_n)_{n\geq 0}$ given by the recurrence $(x_n)_{n\geq 0}, x_0 = 0, x_1 = 1, x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n, \forall n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} \left(\sqrt[n+1]{F_{n+1}x_{n+1}} - \sqrt[n]{F_n x_n} \right) = \frac{\alpha}{e}.$$

Proof. We have $x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n \Leftrightarrow$

$x_{n+2} - (n+2)x_{n+1} = (n+3)(x_{n+1} - (n+1)x_n) \Leftrightarrow y_{n+1} = (n+3)y_n$, where $y_n = x_{n+1} - (n+1)x_n, y_0 = x_1 - x_0 = 1, y_1 = x_2 - 2x_1 = 5 - 2\cdot 1 = 3$. Therefore $y_{k+1} = (k+3)y_k, \forall k \in \mathbb{N}$, so $\prod_{k=0}^{n} y_{k+1} = \prod_{k=0}^{n}(k+3) \cdot \prod_{k=0}^{n} y_k$ which yields successively to $y_{n+1} = y_0 \prod_{k=0}^{n}(k+3) = \frac{(n+3)!}{2}$ or $y_n = \frac{(n+2)!}{2}$. Then we have successively that

$$x_{k+1} - (k+1)x_k = y_k = \frac{(k+2)!}{2} \Leftrightarrow$$

$$\frac{x_{k+1}}{(k+1)!} - \frac{x_k}{k!} = \frac{k+2}{2} = \frac{k}{2} + 1 \Leftrightarrow \sum_{k=0}^{n} \frac{x_{k+1}}{(k+1)!} - \sum_{k=0}^{n} \frac{x_k}{k!} = n + 1 + \frac{1}{2}\sum_{k=0}^{n} k =$$

$$= n + 1 + \frac{n(n+1)}{4} = \frac{(n+1)(n+4)}{4} \Leftrightarrow \frac{x_{n+1}}{(n+1)!} = \frac{(n+1)(n+4)}{4} \Leftrightarrow$$

$\Leftrightarrow x_{n+1} = \frac{(n+1)(n+4)}{4}(n+1)! \Leftrightarrow x_n = \frac{n(n+3)}{4} \cdot n!, \forall n \in \mathbb{N}^*$. We have

$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \frac{1}{\alpha}$, where $\alpha = \frac{\sqrt{5}+1}{2}, \beta = \frac{1-\sqrt{5}}{2}$,

$F_n = \frac{1}{\sqrt{5}}$. Now we obtain that

$$\lim_{n\to\infty} \frac{\sqrt[n]{F_n x_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{F_n x_n}{n^n}} = \lim_{n\to\infty} \frac{F_{n+1}x_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{F_n x_n} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{x_{n+1}}{(n+1)x_n} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{\alpha}{e} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} = \frac{\alpha}{e} \lim_{n\to\infty} \frac{(n+1)(n+4)(n+1)!}{4} \cdot \frac{4}{(n+1)^2(n+3)n!} =$$

$= \frac{\alpha}{e} \cdot 1 = \frac{\alpha}{e}$. We denote $u_n = \frac{\sqrt[n+1]{F_{n+1}x_{n+1}}}{\sqrt[n]{F_n x_n}}$ and we have $\lim_{n\to\infty} u_n = 1$, so

$$\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1 \text{ respectively}$$

$$\lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{F_{n+1}x_{n+1}}{F_n x_n} \cdot \frac{1}{\sqrt[n+1]{F_{n+1}x_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} \cdot \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{L_{n+1}x_{n+1}}} == \alpha \cdot 1 \cdot \frac{e}{\alpha} = e$$

Hence, $\lim_{n\to\infty} (\sqrt[n+1]{F_{n+1}x_{n+1}} - \sqrt[n]{F_n x_n}) = \lim_{n\to\infty} \frac{\sqrt[n]{F_n x_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n =$

$$= \frac{\alpha}{e} \cdot 1 \cdot \ln e = \frac{\alpha}{e}$$

□

Theorem 11. Let the sequence $(x_n)_{n\geq 0}$ given by the recurrence $(x_n)_{n\geq 0}, x_0 = 0, x_1 = 1, x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n, \forall n \in \mathbb{N}$. Then:

$$\lim_{n\to\infty} (\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}} - \sqrt[n]{F_n L_n x_n}) = \frac{\alpha^2}{e}.$$

Proof. We have $x_{n+2} = (2n+5)x_{n+1} - (n^2+4n+3)x_n \Leftrightarrow$

$$x_{n+2} - (n+2)x_{n+1} = (n+3)(x_{n+1} - (n+1)x_n) \Leftrightarrow y_{n+1} = (n+3)y_n$$

where $y_n = x_{n+1} - (n+1)x_n, y_0 = x_1 - x_0 = 1, y_1 = x_2 - 2x_1 = 5 - 2 \cdot 1 = 3$

Therefore, $y_{k+1} = (k+3)y_k, \forall k \in \mathbb{N}$, so $\prod_{k=0}^{n} y_{k+1} = \prod_{k=0}^{n} (k+3) \cdot \prod_{k=0}^{n} y_k$

which yields succesively to $y_{n+1} = y_0 \prod_{k=0}^{n} (k+3) = \frac{(n+3)!}{2}$ or $y_n = \frac{(n+2)!}{2}$

Then we have successively that $x_{k+1} - (k+1)x_k = y_k = \frac{(k+2)!}{2} \Leftrightarrow$

$$\frac{x_{k+1}}{(k+1)!} - \frac{x_k}{k!} = \frac{k+2}{2} = \frac{k}{2} + 1 \Leftrightarrow \sum_{k=0}^{n} \frac{x_{k+1}}{(k+1)!} - \sum_{k=0}^{n} \frac{x_k}{k!} = n + 1 + \frac{1}{2}\sum_{k=0}^{n} k =$$

$$= n + 1 + \frac{n(n+1)}{4} = \frac{(n+1)(n+4)}{4} \Leftrightarrow \frac{x_{n+1}}{(n+1)!} = \frac{(n+1)(n+4)}{4} \Leftrightarrow$$

$$\Leftrightarrow x_{n+1} = \frac{(n+1)(n+4)}{4}(n+1)! \Leftrightarrow x_n = \frac{n(n+3)}{4} \cdot n!, \forall n \in \mathbb{N}^*$$

We have $\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \frac{1}{\alpha}$ and

$\lim_{n\to\infty} \frac{L_n}{L_{n+1}} = \lim_{n\to\infty} \frac{\alpha^n + \beta^n}{\alpha^{n+1} + \beta^{n+1}} = \frac{1}{\alpha}$, where $\alpha = \frac{\sqrt{5+1}}{2}, \beta = \frac{1-\sqrt{5}}{2}$

$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), L_n = \alpha^n + \beta^n$. Now we obtain that

$$\lim_{n\to\infty} \frac{\sqrt[n]{F_n L_n x_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{F_n L_n x_n}{n^n}} = \lim_{n\to\infty} \frac{F_{n+1}L_{n+1}x_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{F_n L_n x_n} =$$

$$= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{L_{n+1}}{L_n} \cdot \frac{x_{n+1}}{(n+1)x_n} \cdot \left(\frac{n}{n+1}\right)^n =$$

$$= \frac{\alpha^2}{e} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} = \frac{\alpha^2}{e} \lim_{n\to\infty} \frac{(n+1)(n+4)(n+1)!}{4} \cdot \frac{4}{(n+1)^2(n+3)n!} = \frac{\alpha^2}{e} \cdot 1 = \frac{\alpha^2}{e}$$

We denote $u_n = \frac{\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}}}{\sqrt[n]{F_nL_nx_n}}$ and we have $\lim_{n\to\infty} u_n = 1$, so $\lim_{n\to\infty} \frac{u_n - 1}{\ln u_n} = 1$

$$\text{respectively } \lim_{n\to\infty} u_n^n = \lim_{n\to\infty} \frac{F_{n+1}L_{n+1}x_{n+1}}{F_nL_nx_n} \cdot \frac{1}{\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}}} =$$

$$= \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{L_{n+1}}{L_n} \lim_{n\to\infty} \frac{x_{n+1}}{(n+1)x_n} \cdot \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}}} = \alpha^2 \cdot 1 \cdot \frac{e}{\alpha^2} = e$$

Hence, $\lim_{n\to\infty} \left(\sqrt[n+1]{F_{n+1}L_{n+1}x_{n+1}} - \sqrt[n]{F_nL_nx_n} \right) = \lim_{n\to\infty} \frac{\sqrt[n]{F_nL_nx_n}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n =$

$$= \frac{\alpha^2}{e} \cdot 1 \cdot \ln e = \frac{\alpha^2}{e}$$

□

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