NEW LIMITS INVOLVING FAMOUS SEQUENCES

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Abstract. In this paper we present some certain limits of sequences.

Theorm 1.

$$\lim_{n\to\infty}\left(\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot e^{x_n}\right)=e^{\gamma}$$

$$\text{where } x_n=\sum_{k=1}^{n}\frac{1}{k}.$$

Proof.

Let $\gamma_n=-\ln n+x_n$ with $\lim\limits_{n\to\infty}\gamma_n=\gamma=$ Euler-Mascheroni constant. We have that

$$a_n=\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot e^{x_n}=\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot n\cdot\frac{e^{x_n}}{n}=\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot n\cdot\frac{e^{x_n}}{e^{\ln n}}=$$

$$(1)\qquad =\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot n\cdot e^{-\ln n+x_n}=\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)\cdot n\cdot e^{\gamma_n}$$

Hence,

$$\lim_{n\to\infty}a_n=e^{\gamma}\cdot\lim_{n\to\infty}\frac{\left(\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}\right)}{\frac{1}{n}}\overset{\text{Cesaro-Stolz}(\frac{0}{0})}{=}e^{\gamma}\cdot\frac{-\frac{1}{(n+1)^2}}{\frac{1}{n+1}-\frac{1}{n}}=e^{\gamma}\cdot\lim_{n\to\infty}\frac{n}{n+1}=e^{\gamma}\cdot 1=e^{\gamma}$$

□

Theorem 2.

$$\lim_{n\to\infty}\frac{1}{(\sqrt[n]{(2n-1)!!})^2}\sum_{k=1}^{n}[(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!})^2]=\frac{e}{2},$$

where we denote by $[x]$ the integer part of x.

Proof.

$$\text{We denote } x_n=\frac{1}{(\sqrt[n]{(2n-1)!!})^2}\sum_{k=1}^{n}[(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!})^2]$$

$$\text{We have } x_n=\left(\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^2\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}[(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!})^2],\text{ so}$$

$$(1)\qquad \lim_{n\to\infty}x_n=\left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}}\right)^2\cdot\lim_{n\to\infty}\sum_{k=1}^{n}[(\sqrt[2k]{k!}+\sqrt[2(k+1)]{(k+1)!})^2]$$

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(2n-1)!!}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} =$$

(2) $$= \lim_{n\to\infty} \frac{n+1}{2n+1}\left(\frac{n+1}{n}\right)^n = \frac{e}{2} \text{ and similarly}$$

(3) $$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

We denote $v_k = \sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}$. We have:

(4) $$\lim_{n\to\infty} \frac{1}{n^1}\sum_{k=1}^{n}[v_k^2] \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1}[v_k^2] - \sum_{k=1}^{n}[v_k^2]}{(n+1)^2 - n^2} = \lim_{n\to\infty} \frac{[v_{n+1}^2]}{2n+1}$$

We have $[v_{n+1}^2] \leq v_{n+1}^2 < [v_{n+1}^2] + 1 \Leftrightarrow \frac{[v_{n+1}^2]}{2n+1} \leq \frac{v_{n+1}^2}{2n+1} < \frac{[v_{n+1}^2]+1}{2n+1}$, so

(5)

$$\lim_{n\to\infty} \frac{[v_{n+1}^2]}{2n+1} = \lim_{n\to\infty} \frac{v_{n+1}^2}{2n+1} = \lim_{n\to\infty} \frac{v_n^2}{2n+1} = \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!} + 2\sqrt{\sqrt[n]{n!}\cdot\sqrt[n+1]{(n+1)!}} + \sqrt[n+1]{(n+1)!}}{2n+1}\right) = \frac{2}{e}$$

$$\text{Hence, } \lim_{n\to\infty} x_n = \frac{e^2}{4}\cdot\frac{2}{e} = \frac{e}{2}.$$

□

Theorem 3.

If $(a_n)_{n\geq 1}$ is a sequence given by $a_n = \sum_{k=1}^{n}\frac{1}{k}$, then

$$\lim_{n\to\infty} e^{-2a_n}\sum_{k=1}^{n}[(\sqrt{k}+\sqrt{k+1})^2] = \frac{2}{e^{2\gamma}},$$

where we denote by $[x]$ the integer part of x.

Proof. We denote

(1) $$x_n = e^{-2a_n}\sum_{k=1}^{n}[(\sqrt{k}+\sqrt{k+1})^2] = \frac{n^2}{e^{2a_n}}\cdot\frac{1}{n^2}\cdot\sum_{k=1}^{n}[(\sqrt{k}+\sqrt{k+1})^2]$$

We have $\frac{n^2}{e^{2a_n}} = \frac{e^{2\ln n}}{e^{2a_n}} = \left(\frac{e^{\ln n}}{e^{a_n}}\right)^2 = \left(\frac{1}{e^{-\ln n + a_n}}\right)^2 = \frac{1}{e^{2\gamma_n}}$, where $\gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$ with

(2)

$\lim_{n\to\infty}\gamma_n = \gamma =$ Euler-Mascheroni constant; so we deduce that $\lim_{n\to\infty}\frac{n^2}{e^{2a_n}} = \frac{1}{e^{2\gamma}} = e^{-2\gamma}$

We have:

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}[(\sqrt{k}+\sqrt{k+1})^2] \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}[(\sqrt{k}+\sqrt{k+1})^2] - \sum_{k=1}^{n}[(\sqrt{k}+\sqrt{k+1})^2]}{(n+1)^2-n^2} =$$

(3) $$= \lim_{n\to\infty}\frac{[(\sqrt{n+1}+\sqrt{n+2})^2]}{2n+1}$$

We have $\frac{[(\sqrt{n+1}+\sqrt{n+2})^2]}{2n+1} \leq \frac{(\sqrt{n+1}+\sqrt{n+2})^2}{2n+1} < \frac{[(\sqrt{n+1}+\sqrt{n+2})^2]}{2n+1} + 1$, so

$$= \lim_{n\to\infty} \frac{[(\sqrt{n+1}+\sqrt{n+2})^2]}{2n+1} = \lim_{n\to\infty} \frac{(\sqrt{n+1}+\sqrt{n+2})^2}{2n+1} =$$

$$= \lim_{n\to\infty} \frac{n+1+n+2+2\sqrt{(n+1)(n+2)}}{2n+1} = 2 \tag{4}$$

From $(1), (2), (3)$ and (4) we obtain $\lim\limits_{n\to\infty} x_n = 2 \cdot e^{-2\gamma} = \dfrac{2}{e^{2\gamma}}$.

□

Theorem 4. If $(a_n)_{n\geq 1}$ is a positive real sequence sequence such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{n^{t+1}a_n} = a \in \mathbb{R}_+^*$ where t is a positive integer number, then

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^{n} [k^t \cdot b] = \frac{e^{t+1}}{a} \cdot \frac{b}{t+1} = \frac{be^{t+1}}{a(t+1)},$$

where $b \in \mathbb{R}$; we denote be $[x]$ the integer part of x.

Proof. We denote

$$x_n = \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^{n} [k^t \cdot b] = \frac{n^{t+1}}{\sqrt[n]{a_n}} \cdot \frac{1}{n^{t+1}} \cdot \sum_{k=1}^{n} [k^t b] \tag{1}$$

$$\text{We have } \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^{t+1}}{\sqrt[n]{a_n}} \cdot \lim_{n\to\infty} \frac{1}{n^{t+1}} \cdot \sum_{k=1}^{n} [k^t b] \tag{2}$$

But, $\lim\limits_{n\to\infty} \dfrac{n^{t+1}}{\sqrt[n]{a_n}} = \lim\limits_{n\to\infty} \sqrt[n]{\dfrac{n^{n(t+1)}}{a_n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim\limits_{n\to\infty} \dfrac{(n+1)^{(n+1)(t+1)}}{a_{n+1}} \cdot \dfrac{a_n}{n^{n(t+1)}} =$

$$= \lim_{n\to\infty} \frac{n^{t+1}a_n}{a_{n+1}} \left(\frac{n+1}{n}\right)^{(n+1)(t+1)} = \frac{e^{t+1}}{a} \text{ and} \tag{3}$$

$$\lim_{n\to\infty} \frac{1}{n^{t+1}} \cdot \sum_{k=1}^{n} [k^t b] \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} [k^t b] - \sum_{k=1}^{n} [k^t b]}{(n+1)^{t+1} - n^{t+1}} = \lim_{n\to\infty} \frac{[(n+1)^t b]}{(n+1)^{t+1} - n^{t+1}} =$$

$$= \lim_{n\to\infty} \frac{[(n+1)^t b] \cdot n^t}{((n+1)^{t+1} - n^{t+1}} = \frac{1}{t+1} \lim_{n\to\infty} \frac{[(n+1)^t b]}{n^t} \tag{4}$$

We have

$[(n+1)^t b] \leq (n+1)^t b < [(n+1)^t b] + 1 \Leftrightarrow \dfrac{[(n+1)^t b]}{n^t} \leq \dfrac{(n+1)^t b}{n^t} < \dfrac{[(n+1)^t b]}{n^t} + \dfrac{1}{n^t}$, then

$$\lim_{n\to\infty} \frac{[(n+1)^t b]}{n^t} = \lim_{n\to\infty} \frac{(n+1)^t b}{n^t} = b \tag{5}$$

Hence, from $(1), (2), (3), (4)$ and (5) we obtain $\lim\limits_{n\to\infty} x_n = \dfrac{e^{t+1}}{a} \cdot \dfrac{b}{t+1} = \dfrac{be^{t+1}}{a(t+1)}$.

□

Theorem 5.

If $(a_n)_{n\geq 1}$ is a positive real sequence given by $a_n = \sum_{k=1}^{n} \frac{1}{k}$, then

$$\lim_{n\to\infty} e^{-2a_n} \sum_{k=1}^{n} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2] = \frac{2}{e^{1+2\gamma}},$$

where we denote by $[x]$ the integer part of x.

Proof. We denote
(1)

$$x_n = e^{-2a_n} \sum_{k=1}^{n} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2] = \frac{n^2}{e^{2a_n}} \cdot \frac{1}{n^2} \cdot \sum_{k=1}^{n} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2]$$

We have $\frac{n^2}{e^{2a_n}} = \frac{e^{2\ln n}}{e^{2a_n}} = \left(\frac{e^{\ln n}}{e^{a_n}}\right)^2 = \left(\frac{1}{e^{-\ln n + a_n}}\right)^2 = \frac{1}{e^{2\gamma_n}}$, where $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with

(2)

$\lim_{n\to\infty} \gamma_n = \gamma =$ Euler-Mascheroni constant; so we deduce that $\lim_{n\to\infty} \frac{n^2}{e^{2a_n}} = \frac{1}{e^{2\gamma}} = e^{-2\gamma}$

We have $\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2] \overset{\text{Cesaro-Stolz}}{=}$

$$\overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2] - \sum_{k=1}^{n} [(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!})^2}{(n+1)^2 - n^2} =$$

(3)

$$= \lim_{n\to\infty} \frac{[(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2]}{2n+1}$$

We have $\frac{[(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2]}{2n+1} \leq \frac{(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2}{2n+1} <$

$$< \frac{[(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2]}{2n+1} + \frac{1}{2n+1}, \text{ so}$$

$$\lim_{n\to\infty} \frac{[(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2]}{2n+1} = \lim_{n\to\infty} \frac{(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!})^2}{2n+1} =$$

(4)

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!} + \sqrt[n+2]{(n+2)!} + 2 \cdot \sqrt[2(n+1)]{(n+1)!} \cdot \sqrt[2(n+2)]{(n+2)!}}{2n+1} = \frac{1}{2}\left(\frac{1}{e} + \frac{2}{e} + \frac{1}{e}\right) = \frac{2}{e}$$

where we used the well-knon fact that $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$

From $(1), (2), (3)$ and (4) we obtain $\lim_{n\to\infty} x_n = e^{-2\gamma} \cdot \frac{2}{e} = \frac{2}{e^{1+2\gamma}}$.

□

Theorem 6.

If $(a_n)_{n\geq}$ is a positive real sequence such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*$, then

$$\text{a) } \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \cdot \sqrt[n^2]{\frac{(n!)^n}{b}} = \frac{1}{a}, \text{ where } b \in \mathbb{R}_+^*;$$

$$\text{b) } \lim_{n\to\infty} \sqrt[n^2]{\frac{(n+1)^n}{n!}} = 1.$$

Proof.

$$\text{a) We have: } \frac{1}{\sqrt[n]{a_n}} \cdot \sqrt[n^2]{\frac{(n!)^n}{b}} = \frac{n}{\sqrt[n]{a_n}} \cdot \frac{\sqrt[n]{n!}}{n} \cdot \frac{1}{\sqrt[n]{b}} \tag{1}$$

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \tag{2}$$

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{a_{n+1}} \cdot \frac{a_n}{n^n} = \lim_{n\to\infty} \frac{na_n}{a_{n+1}} \left(\frac{n+1}{n}\right)^n = \frac{e}{a} \tag{3}$$

From (1), (2) and (3) we obtain $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \cdot \sqrt[n^2]{\frac{(n!)^n}{b}} = \frac{e}{a} \cdot \frac{1}{e} \cdot \frac{1}{1} = \frac{1}{a}$

b) $\sqrt[n^2]{\frac{(n+1)^n}{n!}} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n+1}}{\sqrt[n^2]{n!}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n+1}{\sqrt[n]{n!}}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n}{\sqrt[n]{n!}} \cdot \frac{n}{n+1}} = \lim\limits_{n\to\infty} \sqrt[n]{e \cdot 1} = 1.$

∎

Theorem 7.

If $(s_n)_{n\geq 1}$ is a sequence given by $s_n = -2\sqrt{n} + \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim\limits_{n\to\infty} s_n = s =$

$=$ Ioachimescu constant and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$, be a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*, \lim_{n\to\infty} \frac{b_{n+1}}{b_n\sqrt{n}} = b \in \mathbb{R}_+^*, \text{ then}$$

$$\lim_{n\to\infty} \left(1 + e^{s_n} - e^{s_{n+1}}\right)^{\sqrt[n]{a_n b_n}} = e^{\frac{ab}{4e\sqrt{e}}}$$

Proof. We denote:

$$x_n = \left(1 + e^{s_n} - e^{s_{n+1}}\right)^{\sqrt[n]{a_n b_n}} = \left(\left(1 + e^{s_n} - e^{s_{n+1}}\right)^{\frac{1}{s_n - s_{n+1}}}\right)^{(s_n - s_{n+1}) \cdot \sqrt[n]{a_n b_n}} =$$

$$= \left(\left(1 + e^{s_n} - e^{s_{n+1}}\right)^{\frac{1}{s_n - s_{n+1}}}\right)^{n\sqrt{n}(s_n - s_{n+1}) \cdot \frac{\sqrt[n]{a_n b_n}}{n\sqrt{n}}}$$

and then

$$\lim_{n\to\infty} x_n = e^{\lim_{n\to\infty}(s_n - s_{n+1}) \cdot n\sqrt{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{a_n b_n}}{n\sqrt{n}}} \tag{1}$$

We have: $\lim\limits_{n\to\infty} \frac{\sqrt[n]{a_n b_n}}{a\sqrt{n}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{a_n b_n}{(n\sqrt{n})^n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim\limits_{n\to\infty} \frac{a_{n+1} b_{n+1}}{((n+1)\sqrt{n+1})^{n+1}} \cdot \frac{(n\sqrt{n})^n}{a_n b_n} =$

$$= \lim_{n\to\infty} \frac{a_{n+1}}{a_n n} \cdot \frac{b_{n+1}}{b_n\sqrt{n}} \cdot \left(\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}}\right)^{n+1} = a \cdot b \cdot \frac{1}{e\sqrt{e}} = \frac{ab}{e\sqrt{e}} \tag{2}$$

$$\lim_{n\to\infty} n\sqrt{n}\cdot(s_n - s_{n+1}) = \lim_{n\to\infty} n\sqrt{n}\Big(2\sqrt{n+1} - 2\sqrt{n} - \frac{1}{\sqrt{n+1}}\Big) =$$

$$= \lim_{n\to\infty} \frac{n\sqrt{n}}{\sqrt{n+1}}(2n + 2 - 2\sqrt{n(n+1)} - 1) =$$

(3)

$$= \lim_{n\to\infty} n(2n+1-2\sqrt{n(n+1)}) = \lim_{n\to\infty} n\cdot\frac{(2n+1)^2 - 4n(n+1)}{2n+1+2\sqrt{n(n+1)}} = \lim_{n\to\infty} n\cdot\frac{1}{2n+1+2\sqrt{n(n+1)}} = \frac{1}{4}$$

From (1), (2) and (3) we obtain that $\lim_{n\to\infty} x_n = e^{\frac{ab}{e\sqrt{e}}\cdot\frac{1}{4}} = e^{\frac{ab}{4e\sqrt{e}}}$

□

Theorem 8.

If $m, p \in \mathbb{R}_+^*, m > p$ and $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma =$ Euler - Mascheroni

constant and $(a_n)_{n\geq 1}$ is a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^m a_n} = a \in \mathbb{R}_+^*$, then

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^{m-p}}(\gamma_n - \gamma)^p = \frac{a}{2^p e^m}$$

Proof.

(1) We denote $x_n = \frac{\sqrt[n]{a_n}}{n^{m-p}}(\gamma_n - \gamma)^p = \frac{\sqrt[n]{a_n}}{n^m}(n(\gamma - \gamma_n))^p$

We have:

(2)

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n^m} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^{mn}}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{m(n+1)}}\cdot\frac{n^{mn}}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{n^m a_n}\Big(\frac{n}{n+1}\Big)^{m(n+1)} = \frac{a}{e^m}$$

$$\lim_{n\to\infty} n(\gamma_n - \gamma) = \lim_{n\to\infty} \frac{\gamma_n - \gamma}{\frac{1}{n}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\gamma_{n+1} - \gamma - \gamma_n + \gamma}{\frac{1}{n+1} - \frac{1}{n}} =$$

$$= \lim_{n\to\infty} \frac{\gamma_n - \gamma_{n+1}}{\frac{1}{n} - \frac{1}{n+1}} = \lim_{n\to\infty} \frac{\ln\frac{n+1}{n} - \frac{1}{n+1}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{(n+1)\ln\frac{n+1}{n} - 1}{\frac{1}{n}} = \lim_{\substack{x\to 0\\ x>0}} \frac{(\frac{1}{x}+1)\ln(1+x) - 1}{x} =$$

(3)

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{(x+1)\ln(1+x) - x}{x^2} \overset{\text{L'Hospital}}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(x+1) + 1 - 1}{2x} = \frac{1}{2}\lim_{\substack{x\to 0\\ x>0}} (1+x)^{\frac{1}{x}} = \frac{1}{2}\ln e = \frac{1}{2}$$

From (1), (2) and (3) we obtain $\lim_{n\to\infty} x_n = \frac{a}{e^m}\Big(\frac{1}{2}\Big)^p = \frac{a}{2^p e^m}$

□

Theorem 9. If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n a_n} = a \in \mathbb{R}_+^* \text{ then } \lim_{n\to\infty} \frac{1}{n}\sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}} = \frac{e}{a}$$

Proof.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{a_k}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{k}{\sqrt[k]{a_k}}-\sum_{k=2}^{n}\frac{k}{\sqrt[k]{a_k}}}{(n+1)-n} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{a_{n+1}}} =$$

$$= \lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n} = \lim_{n\to\infty}\frac{na_n}{a_{n+1}}\left(\frac{n+1}{n}\right)^{n+1} = \frac{e}{a}$$

□

Theorem 10.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{(2k-1)!!}} = \frac{e}{2}$$

Proof.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{(2k-1)!!}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{k}{\sqrt[k]{(2k-1)!!}}-\sum_{k=2}^{n}\frac{k}{\sqrt[k]{(2k-1)!!}}}{(n+1)-n} =$$

$$= \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\frac{n}{\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(2n-1)!!}} \overset{\text{Cauchy-D'Alembert}}{=}$$

$$= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(2n+1)!!}\cdot\frac{(2n-1)!!}{n^n} = \lim_{n\to\infty}\frac{n+1}{2n+1}\left(\frac{n+1}{n}\right)^{n+1} = \frac{e}{2}$$

□

Theorem 11.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} = e$$

Proof.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{k}{\sqrt[k]{k!}}-\sum_{k=2}^{n}\frac{k}{\sqrt[k]{k!}}}{(n+1)-n} = \lim_{n\to\infty}\frac{n+1}{\sqrt[n+1]{(n+1)!}} =$$

$$= \lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

□

Theorem 12.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{\sqrt[k]{(k!)^2}} = e^2$$

Proof. We have

$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty}\sqrt[n]{\frac{n^n}{n!}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k^2}{\sqrt[k]{(k!)^2}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}\frac{k^2}{\sqrt[k]{(k!)^2}}-\sum_{k=1}^{n}\frac{k^2}{(k!)^2}}{(n+1)-n} = \lim_{n\to\infty}\frac{(n+1)^2}{\sqrt[n+1]{((n+1)!)^2}} =$$

$$= \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{(n!)^2}} = \left(\lim_{n\to\infty}\frac{n}{\sqrt[n]{n!}}\right)^2 = e^2$$

□

Theorem 13.

If $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma =$ Euler-Mascheroni constant, then

$$\lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} k^{\gamma}} = 1 \text{ and } \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} k^{\pi}} = 1.$$

Proof.

$$\text{Let } a > 0 \text{ and } x_n = \sqrt[n]{\sum_{k=1}^{n} k^a}. \text{ We have that}$$

$$1 < x_n < \sqrt[n]{n \cdot n^a} = \sqrt[n]{n^{a+1}} = (\sqrt[n]{n})^{a+1}, \forall n \geq 2 \text{ so we deduce that}$$
$$1 \leq \lim_{n\to\infty} x_n < (\lim_{n\to\infty} \sqrt[n]{n})^{a+1} = 1^{a+1} = 1.$$

Therefore, $\lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} k^a} = 1, \forall a > 0$. Hence $\lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} k^{\gamma}} = \lim_{n\to\infty} \sqrt[n]{\sum_{k=1}^{n} k^{\pi}} = 1.$

□

Theorem 14.

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{(k!)((2k-1)!!)} = \frac{2}{3e^2}$$

Proof. By Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{(k!)((2k-1)!!)} = \lim_{n\to\infty} \frac{\sum_{k=2}^{n+1} \sqrt[k]{(k!)((2k-1)!!)} - \sum_{k=2}^{n} \sqrt[k]{(k!)((2k-1)!!)}}{(n+1)^3 - n^3} =$$

(1)

$$= \lim_{n\to\infty} \frac{\sqrt[n+1]{((n+1)!)((2n+1)!!)}}{3n^2+3n+1} = \lim_{n\to\infty} \frac{n^2}{3n^2+3n+1} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n}$$

We have:

(2) $$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{1}{e}$$

(3)

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \lim_{n\to\infty} \frac{2n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \frac{2}{e}$$

From (1), (2) and (3) we obtain:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{(k!)((2k-1)!!)} = \frac{1}{3} \cdot \frac{1}{e} \cdot \frac{2}{e} = \frac{2}{3e^2}$$

□

Theorem 15. If $(a_n)_{n\geq 1}$ is a positive real sequence such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n a_n} = a \in \mathbb{R}_+^*, \text{ then } \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{a_k (k!)} = \frac{a}{3e^2}$$

Proof. By Cesaro-Stolz theorem we have:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{a_k(k!)} = \lim_{n\to\infty} \frac{\sum_{k=2}^{n+1} \sqrt[k]{a_k(k!)} - \sum_{k=2}^{n} \sqrt[k]{a_k(k!)}}{(n+1)^3 - n^3} =$$

$$(1) \qquad \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}((n+1)!)}}{3n^2+3n+1} = \lim_{n\to\infty} \frac{n^2}{3n^2+3n+1} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n}$$

We have:

$$(2) \qquad \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

(3)

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{na_n} \left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}$$

From $(1), (2)$ and (3) we obtain:

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=2}^{n} \sqrt[k]{a_k(k!)} = \frac{a}{3e^2}$$

□

Theorem 16. If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^* \text{ and } \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^*, \text{ then } \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}} = \frac{b}{a}$$

Proof. We have:

$$(1) \quad \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \sum_{k=2}^{n} \frac{1}{k} b_k^{\frac{1}{k}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} \cdot \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} \cdot \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}}$$

(2)

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{na_n} \left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}$$

$$(3) \qquad \text{and similarly } \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \frac{b}{e}$$

(4)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \frac{1}{k} b_k^{\frac{1}{k}} - \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}}}{n+1-n} = \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{n+1} = \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \frac{b}{e}$$

From $(1), (2), (3)$ and (4) we obtain:

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}} \sum_{k=1}^{n} \frac{1}{k} b_k^{\frac{1}{k}} = \frac{e}{a} \cdot \frac{b}{e} = \frac{b}{a}$$

□

Theorem 17. If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^* \text{ and } \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^*, \text{ then}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n!)^3}} \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}} = \frac{abe}{3}$$

Proof. We have:

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n!)^3}} \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}} = \lim_{n\to\infty} \left(\frac{n}{\sqrt[n]{n!}}\right)^3 \cdot \frac{1}{n^3} \cdot \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}} =$$

$$(1) \qquad = \lim_{n\to\infty} \left(\frac{n}{\sqrt[n]{n!}}\right)^3 \cdot \lim_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}}$$

(2)

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{n!}} \overset{\text{Cauchy-D'Alembert}}{=} \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

(3)

$$\lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{na_n} \left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{3}$$

$$(4) \qquad \text{and similalry } \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \frac{b}{e}$$

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1} (a_k b_k)^{\frac{1}{k}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} (a_k b_k)^{\frac{1}{k}} - \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}}}{(n+1)^3 - n} =$$

$$= \lim_{n\to\infty} \frac{n^2}{3n^2+3n+1} \cdot \frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{n^2} =$$

$$= \lim_{n\to\infty} \frac{1}{3} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{n^2} = \frac{1}{3} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}}}{(n+1)} \cdot \frac{\sqrt[n+1]{b_{n+1}}}{(n+1)} \cdot \left(\frac{n+1}{n}\right)^2 =$$

$$(4) \qquad = \frac{1}{3} \lim_{n\to\infty} \frac{\sqrt[n]{a_n}}{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} \cdot \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{3} \cdot \frac{a}{e} \cdot \frac{b}{e} \cdot 1 = \frac{ab}{3e^2}$$

From $(1), (2), (3)$ and (4) we obtain:

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(n!)^3}} \sum_{k=1}^{n} (a_k b_k)^{\frac{1}{k}} = e^3 \cdot \frac{ab}{3e^2} = \frac{abe}{3}.$$

□

Therorem 18. If $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ are positive real sequences such that:

$$\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*, \lim_{n\to\infty} \frac{b_{n+1}}{nb_n} = b \in \mathbb{R}_+^* \text{ and } \lim_{n\to\infty} \frac{c_{n+1}}{nc_n} = c \in \mathbb{R}_+^*, \text{ then}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n^3}} \sum_{k=1}^{n} (b_k c_k)^{\frac{1}{k}} = \frac{bce}{3a^3}$$

Proof. We have:

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n^3}}\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{a_n^3}}\right)^3\cdot\frac{1}{n^3}\cdot\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}=$$

(1) $$=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{a_n}}\right)^3\cdot\lim_{n\to\infty}\frac{1}{n^3}\cdot\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}$$

(2)
$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}\overset{\text{Cauchy-D'Alembert}}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}=\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\left(\frac{n+1}{n}\right)^{n+1}=\frac{e}{a}$$

(3)
$$\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{b_n}{n^n}}=\lim_{n\to\infty}\frac{b_{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{b_n}=\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}\left(\frac{n}{n+1}\right)^{n+1}=\frac{b}{e}$$

(4) $$\text{and similarly }\lim_{n\to\infty}\frac{\sqrt[n]{c_n}}{n}=\frac{c}{e}$$

$$\lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}\overset{\text{Cesaro-Stolz}}{=}\lim_{n\to\infty}\frac{\sum_{k=1}^{n+1}(b_kc_k)^{\frac{1}{k}}-\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}}{(n+1)^3-n}=$$

$$=\lim_{n\to\infty}\frac{n^2}{3n^2+3n+1}\cdot\frac{\sqrt[n+1]{b_{n+1}c_{n+1}}}{n^2}=$$

$$=\frac{1}{3}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}c_{n+1}}}{n^2}=\frac{1}{3}\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{b_{n+1}}}{(n+1)}\cdot\frac{\sqrt[n+1]{c_{n+1}}}{(n+1)}\cdot\left(\frac{n+1}{n}\right)^2=$$

(4) $$=\frac{1}{3}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{b_n}}{n}\cdot\lim_{n\to\infty}\frac{\sqrt[n]{c_n}}{n}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2=\frac{1}{3}\cdot\frac{b}{e}\cdot\frac{c}{e}\cdot 1=\frac{bc}{3e^2}$$

From $(1),(2),(3)$ and (4) we obtain:

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n^3}}\sum_{k=1}^{n}(b_kc_k)^{\frac{1}{k}}=\frac{e^3}{a^3}\cdot\frac{bc}{3e^2}=\frac{bce}{3a^3}$$

□

Theorem 19. If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are positive real sequences such that $\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in\mathbb{R}_+^*$, $\lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in\mathbb{R}_+^*$, then $\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n^3}}\sum_{k=1}^{n}((2k-1)!!b_k)^{\frac{1}{k}}=\frac{2be}{a^3}$

Proof. We have:

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n^3}}\sum_{k=1}^{n}((2k-1)!!b_k)^{\frac{1}{k}}=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{a_n^3}}\right)^3\cdot\frac{1}{n^3}\cdot\sum_{k=1}^{n}((2k-1)!!b_k)^{\frac{1}{k}}=$$

(1) $$=\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{a_n}}\right)^3\cdot\lim_{n\to\infty}\frac{1}{n^3}\cdot\sum_{k=1}^{n}((2k-1)!!b_k)^{\frac{1}{k}}$$

(2)
$$\lim_{n\to\infty}\frac{n}{\sqrt[n]{a_n}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{a_n}}\overset{\text{Cauchy-D'Alembert}}{=}\lim_{n\to\infty}\frac{(n+1)^{n+1}}{a_{n+1}}\cdot\frac{a_n}{n^n}=\lim_{n\to\infty}\frac{na_n}{a_{n+1}}\left(\frac{n+1}{n}\right)^{n+1}=\frac{e}{a}$$

(3) and similarly $\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} = \frac{b}{e}$

(4)

$$\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{n^n}} = \lim_{n\to\infty} \frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!} = \lim_{n\to\infty} \frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e}$$

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}((2k-1)!!b_k)^{\frac{1}{k}} \overset{\text{Cesaro-Stolz}}{=} \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1}((2k-1)!!b_k)^{\frac{1}{k}} - \sum_{k=1}^{n}(2k-1)!!b_k)^{\frac{1}{k}}}{(n+1)^3 - n} =$$

$$= \lim_{n\to\infty} \frac{n^2}{3n^2+3n+1} \cdot \frac{\sqrt[n+1]{(2n+1)!!b_{n+1}}}{n^2} = \frac{1}{3} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{(2n+1)!!b_{n+1}}}{n^2} =$$

$$= \frac{1}{3} \cdot \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}}}{(n+1)} \cdot \frac{\sqrt[n+1]{(2n+1)!!}}{(n+1)} \cdot \left(\frac{n+1}{n}\right)^2 =$$

(5) $$= \frac{1}{3}\lim_{n\to\infty} \frac{\sqrt[n]{b_n}}{n} \cdot \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!1}}{n} \cdot \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{1}{3} \cdot \frac{b}{e} \cdot \frac{2}{e} \cdot 1 = \frac{2b}{3e^2}$$

From $(1), (2), (3), (4)$ and (5) we obtain:

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n^3}} \sum_{k=1}^{n} (b_k c_k)^{\frac{1}{k}} = \frac{e^3}{a^3} \cdot \frac{2b}{3e^2} = \frac{2be}{3a^3}$$

□

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