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If $n, k \in \mathbb{N}, n \geq k$ and $f_k: \mathbb{R} \to \left[0, \frac{n}{n+k-1}\right]$ continuous function. Prove that:

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(k)\cdot \sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right) < \log 2$$

Proposed by Florică Anastase-Romania

Solution 1 by Nassim Nicholas Taleb-New York-USA, Solution 2 by proposer

Solution 1 by Nassim Nicholas Taleb-New York-USA

Determine the summand $S(n,k) = f(n,k)\left(\frac{(1-k)f(n,k)}{n} - f(n,k) + 1\right)^{\frac{1}{k+n-1}}$

$S(\cdot)$ –is maximum for $f(n,k) = \frac{n}{n+k-1} - \epsilon, 0 < \epsilon < \frac{n}{n+k-1}$.

Maximizing, we have $\frac{\partial S(n,k)}{\partial \epsilon} = 0$ for $\epsilon = \frac{n}{(k+n-1)(k+n)}$. Allora

$$\sum_{k=1}^{n} S^{Max}(n,k) = \sum_{k=1}^{n}\left(\frac{1}{k+n}\right)^{\frac{k+n}{-1+k+n}} \leq \sum_{k=1}^{n}\left(\frac{1}{k+n}\right) = \psi^{(0)}(2n+1) - \psi^{(0)}(n+1),$$

Where ψ –is the polygamma function. Allora

$$\lim_{n\to\infty}\left(\psi^{(0)}(2n+1) - \psi^{(0)}(n+1)\right) = \log 2$$

Solution 2 by proposer

$$\frac{n}{n+k}=\frac{(n+k-1)f(k)+n-(n+k-1)f(k)}{n+k}\overset{AM-GM}{\geq}$$

$$\geq \sqrt[n+k]{f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big)}\Leftrightarrow$$

$$f^{n+k-1}(k)\big(n-(n+k-1)f(k)\big)\leq\left(\frac{n}{n+k}\right)^{n+k-1}\cdot\frac{n}{n+k}\Leftrightarrow$$

$$f(k)\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\leq\frac{n}{n+k}$$

$$\frac{f(k)}{n}\cdot\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\leq\frac{1}{n}\cdot\frac{n}{n+k}\Leftrightarrow$$

$$\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(n)+\frac{1-k}{n}f(n)}\leq$$

$$\leq\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{\frac{n+k}{n}\big(n-(n+k-1)f(k)\big)}\leq\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n}{n+k}=$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\sum_{k=1}^{n}\frac{n}{n+k}=\int_0^1\frac{1}{x+1}dx=log2$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}f(k)\cdot\sqrt[n+k-1]{1-f(k)+\frac{1-k}{n}f(k)}\right)<log2$$