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If $0 < a \le b, f:[a,b] \to [0,\infty), f$ –continuous function, then:

$$2(b-a)\int_a^b f(x)dx \ge \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco, Solution 2 by Remus Florin Stanca-Romania

Solution 1 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\left(\int_a^b \sqrt{f(x)}dx\right)^2 + \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy =$$

$$= \int_a^b\int_a^b \sqrt{f(x)}\sqrt{f(y)}dxdy + \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy =$$

$$= \int_a^b\int_a^b \left[\sqrt{f(x)f(y)} + \sqrt{\frac{1}{2}\left(f^2(x)+f^2(y)\right)}\right]dxdy \overset{CBS}{\le}$$

$$\le \int_a^b\int_a^b \sqrt{2\left[f(x)f(y)+\frac{1}{2}\left(f^2(x)+f^2(y)\right)\right]}dxdy = \int_a^b\int_a^b \left(f(x)+f(y)\right)dxdy =$$

$$2(b-a)\int_a^b f(x)dx$$

Solution 2 by Remus Florin Stanca-Romania

$$2(b-a)\int_a^b f(x)dx \geq \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{f^2(x)+f^2(y)}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2 \Leftrightarrow$$

$$2(b-a)\int_a^b f(x)dx \geq \frac{1}{\sqrt{2}}\int_a^b\int_a^b \sqrt{\frac{f^2(x)+f^2(y)}{2}}dxdy + \left(\int_a^b \sqrt{f(x)}dx\right)^2 \Leftrightarrow$$

$$\int_a^b\int_a^b \big(f(x)+f(y)\big)dxdy \geq \int_a^b\int_a^b \sqrt{\frac{f^2(x)+f^2(y)}{2}}dxdy + \int_a^b\int_a^b \sqrt{f(x)}\sqrt{f(y)}dxdy$$

We need to prove that: $a+b \geq \sqrt{\frac{a^2+b^2}{2}} + \sqrt{ab} \Leftrightarrow \frac{a+b}{2} \geq \frac{1}{2}\sqrt{\frac{a^2+b^2}{2}} + \frac{1}{2}\sqrt{ab}; (1)$

Let $f:\mathbb{R}_+ \to \mathbb{R}$ such that: $f(x)=\sqrt{x}, f'(x)=\frac{1}{2}x^{-\frac{1}{2}}, f''(x)=-\frac{1}{4}x^{-\frac{3}{2}} \leq 0 \Rightarrow$

f –concave$\Rightarrow \forall t_1, t_2 \in \left(0, \frac{\pi}{2}\right)$ such that $t_1+t_2=1$ and $\forall x_1, x_2 \in I$:

$$t_1 f(x_1) + t_2 f(x_2) \leq f(t_1 x_2 + t_2 x_2); (JENSEN)$$

$$\frac{1}{2}\sqrt{\frac{a^2+b^2}{2}} + \frac{1}{2}\sqrt{ab} \leq \sqrt{\frac{a^2+b^2+2ab}{4}} = \frac{a+b}{2} \Rightarrow (1) \text{ is true.}$$

Note by editor:

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