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Evaluate the integral in a closed –form:

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx$$

Proposed by Srinivasa Raghava-AIRMC-India

Solution 1 by Rana Ranino-Setif-Algerie, Solution 2 by Samar Das-India,
Solution 3 by Timson Azeez Folorunsho-Nigeria

Solution 1 by Rana Ranino-Setif-Algerie

$$\Omega=\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx=\int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx+$$

$$+\left[6\sqrt{2}log\left(2sin\left(\frac{3x}{4}\right)+1\right)-4log(1+4cosx)\right]_0^{\frac{\pi}{3}}$$

$$\Omega=6\sqrt{2}log(1+\sqrt{2})+4log\left(\frac{5}{3}\right)+\underbrace{\int_0^{\frac{\pi}{3}}\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}dx}_{I}$$

$$I=\int_0^{\frac{\pi}{3}}\frac{\sin^2 x}{\cos^2 x\cos^3\left(\frac{x}{2}\right)}dx=4\int_0^{\frac{\pi}{3}}\frac{\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2 x\cos^2\left(\frac{x}{2}\right)}dx=$$

$$=4\int_0^{\frac{\pi}{3}}\frac{\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\left(1-\sin^2\left(\frac{x}{2}\right)\right)\left(1-2\sin^2\left(\frac{x}{2}\right)\right)^2}$$

$$I\overset{t=\sin\frac{x}{2}}{=}8\int_0^{\frac{1}{2}}\frac{t^2}{(1-t^2)(1-2t^2)^2}dt=8\int_0^{\frac{1}{2}}\left(\frac{2}{2t^2-1}+\frac{1}{(2t^2-1)^2}-\frac{1}{t^2-1}\right)dt=$$

$$=8\left[\frac{3\sqrt{2}}{8}\log\left(\frac{\sqrt{2}-2t}{2t+\sqrt{2}}\right)+\frac{t}{2(1-2t^2)}+\frac{1}{2}\log\left(\frac{1+t}{1-t}\right)\right]_0^{\frac{1}{2}}=$$

$$=3\sqrt{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)+4+4\log 3=4-6\sqrt{2}\log(1+\sqrt{2})+4\log 3=$$

$$=6\sqrt{2}\log(1+\sqrt{2})+4\log\left(\frac{5}{3}\right)+4-6\sqrt{2}\log(1+\sqrt{2})+4\log 3=4(1+\log 5)$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=4(1+\log 5)$$

Solution 2 by Samar Das-India

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=$$

$$=\int_0^{\frac{\pi}{3}}\left(\frac{4\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\left(1-\sin^2\left(\frac{x}{2}\right)\right)\left(1-2\sin^2\left(\frac{x}{2}\right)\right)^2}+\frac{32\sin\frac{x}{2}\cos\frac{x}{2}}{5-8\sin^2\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{2}\right)}{1+2\sin\left(\frac{3x}{2}\right)}\right)dx=$$

$$\overset{y=\sin\frac{x}{2}}{=}\int_0^{\frac{1}{2}}\frac{8y^2}{(1-y^2)(1-2y^2)^2}dy+64\int_0^{\frac{1}{2}}\frac{ydy}{5-8y^2}+6\sqrt{2}\log\left|1+2\sin\left(\frac{3x}{4}\right)\right|_0^{\frac{\pi}{3}}=$$

$$=8\int_0^{\frac{1}{2}}\frac{1}{1-2y^2}\left(\frac{1}{y^2-1}-\frac{1}{2y^2-1}\right)dy-\frac{64}{16}\int_0^{\frac{1}{2}}\frac{ydy}{5-8y^2}+6\sqrt{2}\log(1+\sqrt{2})=$$

$$= -8\int_0^{\frac{1}{2}}\left(\frac{1}{y^2-1}-\frac{1}{y^2-\frac{1}{2}}\right)dy+\int_0^{\frac{1}{2}}\left(\frac{1}{y-\frac{1}{\sqrt{2}}}-\frac{1}{y+\frac{1}{\sqrt{2}}}\right)^2 dy$$

$$-4\left(\log\left|5-\frac{8}{4}\right|-\log 5\right)+6\sqrt{2}\log\left(1+\sqrt{2}\right)=$$

$$= -4\log\left|\frac{y-1}{y+1}\right|_0^{\frac{1}{2}}+\frac{8\sqrt{2}}{2}\log\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\frac{dy}{\left(y-\frac{1}{\sqrt{2}}\right)^2}+\int_0^{\frac{1}{2}}\frac{dy}{\left(y+\frac{1}{\sqrt{2}}\right)^2}-$$

$$-2\int_0^{\frac{1}{2}}\frac{dy}{y^2-\left(\frac{1}{\sqrt{2}}\right)^2}-4\log\left(\frac{3}{5}\right)+6\sqrt{2}\log\left(1+\sqrt{2}\right)=$$

$$= -4\log\left(\frac{1}{3}\right)-4\log\left(\frac{3}{5}\right)+4\sqrt{2}\log\left|\frac{\frac{1}{2}-\frac{1}{\sqrt{2}}}{\frac{1}{2}+\frac{1}{\sqrt{2}}}\right|+\left(\frac{1}{y-\frac{1}{\sqrt{2}}}\right)_0^{\frac{1}{2}}+2\cdot\frac{\sqrt{2}}{2}\log\left|\frac{y-\frac{1}{\sqrt{2}}}{y+\frac{1}{\sqrt{2}}}\right|_0^{\frac{1}{2}}$$

$$+6\sqrt{2}\log\left(1+\sqrt{2}\right)$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=4(1+\log 5)$$

Solution 3 by Timson Azeez Folorunsho-Nigeria

$$\int_0^{\frac{\pi}{3}}\left(\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}+\frac{16\sin x}{4\cos x+1}\right)dx=$$

$$=\int_0^{\frac{\pi}{3}}\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}dx+\int_0^{\frac{\pi}{3}}\frac{9\sqrt{2}\cos\left(\frac{3x}{4}\right)}{2\sin\left(\frac{3x}{4}\right)+1}dx+\int_0^{\frac{\pi}{3}}\frac{16\sin x}{4\cos x+1}dx=I_1+I_2+I_3$$

$$I_1=\int_0^{\frac{\pi}{3}}\frac{\tan^2 x}{\cos^3\left(\frac{x}{2}\right)}dx=\int_0^{\frac{\pi}{3}}\frac{\sin^2 x}{\cos^2 x\cos^3\left(\frac{x}{2}\right)}dx\overset{y=\frac{x}{2}}{\cong}2\int_0^{\frac{\pi}{6}}\frac{\sin^2(2y)dy}{\cos^2(2y)\cos^3 y}=$$

$$=8\int_0^{\frac{\pi}{6}}\frac{sin^2y(1-sin^2y)cosy}{(1-2sin^2y)^2(1-sin^2y)^2}dy=8\int_0^{\frac{\pi}{6}}\frac{cosysin^2y}{(1-2sin^2y)(1-sin^2y)}dy\overset{u=siny}{\triangleq}$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{2}{2u^2-1}+\frac{1}{(2u^2-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}u-1}-\frac{1}{\sqrt{2}u+1}+\frac{1}{(2u^2-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=8\int_0^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}u-1}-\frac{1}{\sqrt{2}u+1}+\frac{1}{4(\sqrt{2}u+1)}+\frac{1}{4(\sqrt{2}u+1)^2}-\frac{1}{4(\sqrt{2}u-1)}\right.$$

$$\left.+\frac{1}{4(\sqrt{2}u-1)^2}+\frac{1}{2(u+1)}-\frac{1}{2(u-1)}\right)du=$$

$$=3\sqrt{2}log\left|\frac{\sqrt{2}-2}{\sqrt{2}+2}\right|+\frac{1}{2}log3+5$$

$$I_2=\int_0^{\frac{\pi}{3}}\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}dx\overset{v=\frac{3x}{4}}{\triangleq}\frac{4}{3}\int_0^{\frac{\pi}{4}}\frac{9\sqrt{2}cosv}{2sinv+1}dv\overset{y=2sinv+1}{\triangleq}$$

$$=\frac{4}{3}\int_1^{1+\sqrt{2}}\frac{9\sqrt{2}}{2y}dy=6\sqrt{2}\int_1^{1+\sqrt{2}}\frac{dy}{y}=6\sqrt{2}log(1+\sqrt{2})$$

$$I_3=\int_0^{\frac{\pi}{3}}\frac{16sinx}{4cosx+1}dx\overset{y=4xosx+1}{\triangleq}-4\int_5^3\frac{dy}{y}=4log\left(\frac{5}{3}\right)$$

Therefore,

$$\int_0^{\frac{\pi}{3}}\left(\frac{tan^2x}{cos^3\left(\frac{x}{2}\right)}+\frac{9\sqrt{2}cos\left(\frac{3x}{4}\right)}{2sin\left(\frac{3x}{4}\right)+1}+\frac{16sinx}{4cosx+1}\right)dx=4(1+log5)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.