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ROMANIAN MATHEMATICAL MAGAZINE
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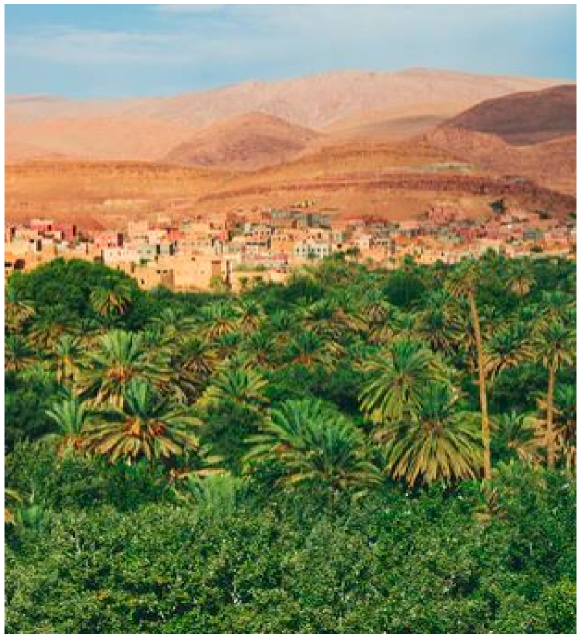

Prove or disprove:

$$\int_0^\infty \frac{\sin(a\pi x)}{(1-x^2)(2^2-x^2)\cdot\ldots\cdot(n^2-x^2)}\frac{dx}{x}=\frac{\pi 2^{2n-1}}{(n!)^2}(1-(-1)^a)$$

Proposed by Ghazaly Abiodun-Nigeria

Solution by Syed Shahabudeen-India

Let us check for $a\in 2k+1,k\in\mathbb{N}$ and $n=2$.

$$LHS=\int_0^\infty \frac{\sin(a\pi x)}{x(1-x^2)(4-x^2)}dx=$$

$$=\lim_{z\to\infty}\int_0^z \sin(a\pi x)\left(\frac{1}{4x}-\frac{1}{6(x+1)}+\frac{1}{6(1-x)}+\frac{1}{24(2-x)}+\frac{1}{24(x+2)}\right)dx=$$

$$=\lim_{z\to\infty}\left(\frac{1}{4}\int_0^{a\pi z}\frac{\sin t}{t}dt+\frac{1}{6}\int_{a\pi}^{a\pi(1+z)}\frac{\sin t}{t}dt-\frac{1}{6}\int_{a\pi}^{a\pi(1-z)}\frac{\sin t}{t}dt-\frac{1}{24}\int_{2a\pi}^{a\pi(2-z)}\frac{\sin t}{t}dt\right.$$

$$\left.+\frac{1}{24}\int_{2a\pi}^{a\pi(2+z)}\frac{\sin t}{t}dt\right)=$$

$$= \lim_{z\to\infty} \frac{1}{24}(6Si(a\pi z) + 4Si(a\pi + \pi z) - 4Si(a\pi - \pi z) - Si(2a\pi - 2z) + Si(2a\pi + 2z)) = \frac{2^3\pi}{4!} = \frac{\pi}{3}$$

If we check the RHS then it gives: $RHS = \frac{2^4\pi}{(2!)^2} \Rightarrow LHS \neq LHS$

If we observe the pattern in LHS for different values of $n \in \mathbb{N}$ **it should be**

$$\frac{\pi 2^{2n-1}}{(n!)^2}(1 - (-1)^a)$$

Therefore,

$$\int_0^\infty \frac{\sin(a\pi x)}{(1-x^2)(2^2-x^2)\cdot\ldots\cdot(n^2-x^2)}\frac{dx}{x} = \frac{\pi 2^{2n-2}}{(2n!)^2}(1-(-1)^a)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.